Introduction

In this thesis we characterize the primitive ideal space of $C^*(G, \Omega)$, where G is a connected, simply connected nilpotent Lie group, and Ω is a locally compact separable Hausdorff space upon which G acts by bicontinuous automorphisms. We are use the space of continuous functions on Ω vanishing at infinity as a commutative C^* algebra upon which G acts, denoted $C_0(\Omega)$.

We require all stability subgroups of points in Ω be connected. We also establish some results on trace class operators related to these algebras.

First we give some general information; we give a section-by-section breakdown of this thesis a little later in this introduction.

The point of this thesis is to show that the orbit method of Kirillov has a natural extension to transformation group C^* algebras which gives information about the primitive ideal spaces of these algebras, as well as providing formulas for characters of these algebras.

The main result of this thesis concerns the case where a connected simply-connected nilpotent Lie group acts on a locally compact separable Hausdorff space Ω in a "nilpotent" fashion. Several reasonable restrictions on this "nilpotent" action are 1) all stability subgroups are connected, and 2) all orbits of G in Ω are locally closed, which is the same as saying that Ω/G is a T_0 space. In Kirillov's paper, [21], he had the hypothesis of connectivity, and the second is a very common "niceness" hypothesis of the orbit space of transformation group C^* algebras.

Let $\mathfrak{g}$ be the Lie algebra of a nilpotent Lie group G and $\mathfrak{g}^*$ be its dual space. Using the Mackey machine developed Takesaki in [25] as well as by Green in [18] for transformation group C^* algebras and some of the ideas employed by Dana Williams in [26], we extend the orbit method of Kirillov to the transformation group C^* algebra setting: given a T_0 orbit space Ω/G we find a map from a quotient space of $\mathfrak{g}^* \times \Omega$ to $\mathrm{Prim}(C^*(G,\Omega))$, which is a homeomorphism to the primitive ideal space of $C^*(G,\Omega)$. We comment that this is also a homeomorphism to the space of irreducible representations of $C^*(G,\Omega)$; we will have more to say about this later. In the case where Ω is a single point this relation restricts to Kirillov's coadjoint equivalence relation on $\mathfrak{g}^*$, which is $\mathfrak{g}^*/G$, which is a homeomorphism to primitive ideal space of G, and by Fell, [11], this is the same as hte primitve ideal space of $C^*(G)$; this relation is described in more detail later in this thesis.

Finally in the third chapter we show that there is a character formula for the representations of $C^*(G,\Omega)$ which generalizes the Kirillov character formula for nilpotent Lie groups.

In [9], Effros and Hahn raised an important question: When is every primitive ideal of a tranformation group C^* algebra "induced" in an appropriate sense from an irreducible representation of a subgroup and a point evaluation of Ω? This question was partially settled by Gootman and Rosenberg in [17], and their work does include all of our examples. However, parameterizing and determining the topology of this primitive ideal space remained an open question.

In [26], under the hypothesis that the group G is abelian, and denoting the set of characters of G as $\widehat{G}$, Dana Williams provides a natural map from a quotient space of $\widehat{G}\times\Omega$ to $\mathrm{Prim}(C^*(G,\Omega))$ which is a homeomorphism.

In [7], Siegfried Echterhoff describes the topology of $\mathrm{Prim}(C^*(G,A))$ (where A need not be an abelian C^* algebra), subject to continuity of the map from A to the closed subgroups of G by $a \mapsto G_a$, the stabilizer of a. We comment that this continuity hypothesis is fairly stringent.

In the case where Ω is a single point, we are dealing with a group C^* algebra. It was proven by Fell in [11] that this is the same as characterizing the representation space of G, and in the case of a connected, simply connected nilpotent Lie group this was solved by Kirillov (in [21]) and Brown (in [1]), also see [19] for a concise proof of Brown's result. We add that Kirillov used the "orbit method" to find a continuous 1-1 onto map from $\mathfrak{g}^*/G$ to $\widehat{G}$, where $\mathfrak{g}^*/G$ denotes the dual space of $\mathfrak{g}$ mod'ed out by an equivalence relation of G acting on $\mathfrak{g}^*$ and identifying elements of the same orbit. We will have more to say about the orbit method later on.

An important part of the harmonic analysis of transformation group C^* algebras is the theory of characters. In the type one situation (see below for definitions), it is known that there is a large number of elements $a \in C^*(G,\Omega)$ such that for any irreducible representation π of $C^*(G,\Omega)$ we have $\pi(a)$ is a trace-class operator. Given an irreducible representation π of $C^*(G,\Omega)$ that "lives" (more on this later) over a G-orbit $\overline{G \cdot x}$ in Ω we will characterize a linear subalgebra $\mathcal{A}_{G\cdot x} \subseteq C_0(G,\Omega)$ such that for $a \in \mathcal{A}_{G\cdot x}$, $\pi(a)$ is a trace-class operator. We also give a generalization of the Kirillov character formula for $\mathrm{Tr}(\pi(a))$, $a \in \mathcal{A}_\pi$.

Theorems 200 and 197 were first proven by Kirillov in [21].

We here give an outline of this paper. Chapter one consists of preliminaries, many of which are used in both chapters 2 and 3; there is one important result in the last section. Chapter one has eleven subsections:

First in section 1.1 we give some basic information about C^* algebras and their representations, followed by descriptions of the topologies on their representation and ideal spaces in section 1.2. In section 1.3 we introduce some of the basic information about nilpotent Lie algebras and groups that will be needed for this paper. Section 1.4 is dedicated to giving a topology, originally defined by Fell, upon the closed subsets of an arbitrary locally compact space and also giving several results particular to nilpotent Lie groups and separable locally compact transformation groups. Section 1.5 is where we first introduce information about unitary representations of a group G and introduce induced representations. Following this in section 1.6 we describe the irreducible representations of a connected, simply-connected nilpotent Lie group, and section 1.7 describes the topology on the space of irreducible representations of an arbitrary locally compact group, with the main focus the case of simply connected nilpotent Lie groups. In section 1.8, a short section, we describe the topology on arbitrary representation spaces of locally compact groups and give several more results that will be needed about induced representations of nilpotent Lie groups. Section 1.9 is dedicated to describing a topology, first introduced by Fell, upon pairs $\langle H, T\rangle$, where H is an arbitrary closed subgroup of a "supergroup" G and T is an arbitrary representation of H. Section 1.10 is dedicated to induced representations and ideals of transformation group C^* algebras. We give two different presentations of this, as we need the second for a result that was proven in the course of this research that isn't really needed for this thesis, but we decided to include anyway, as we may need it for a future abstraction of some of the results. Finally in section 1.11 we give a topology upon representations of C^* algebras arising from subgroups of our nilpotent Lie group G and give one important result that will be needed in chapter 2. Some of the sections in chapter one logically overlap, but we felt that each topic deserved to be handled separately. In chapter 2 we describe the topology of $\mathrm{Prim}(C^*(G,\Omega))$ given that G is a connected, simply connected nilpotent Lie group, given the hypothesis that Ω/G is a T_0 space. In this section we also give a result (Proposition 183) relating to kernels of induced representations of $C^*(G,\Omega)$ that is not specifically needed for this paper, as it is trivial when Ω/G is a T_0 space, but we include it as we may need it for a future abstraction of the results to the case where Ω/G is non-T_0. We comment that we prove this in some abstraction, as the result does not depend upon G being a nilpotent Lie group. The ultimate result of this section abstracts the work of Kirillov and Brown in [21] and [1], as well as expanding upon the work of Dana Williams in [26]. Chapter 3 is devoted to giving a character theory of $C^*(G,\Omega)$ given that G is a connected, simply-connected nilpotent Lie group, and is divided into four subsections. Section 3.1 is devoted to some preliminary information about trace-class operators in general. In section 3.2 we describe the Schwartz functions on a nilpotent Lie group G; these are important in the character theory of such groups. In section 3.3 we describe some operators arising from $C^*(G)$ which are trace class, we comment that all references here may be found in [3], and the original work was done by Kirillov in [21]. Finally in section 3.4, we let G be our usual connected, simply-connected nilpotent Lie group, and let x be a fixed point in Ω, with its associated G-orbit $G \cdot x$. We let L be an arbitrary irreducible representation which "lives" (more on what this means later) over $\overline{G \cdot x}$ and characterize a linear subalgebra $\mathcal{A}_{G\cdot x}$ of $C^*(G,\Omega)$ upon which $L(a)$ is a trace-class operator for each $a \in \mathcal{A}_{G\cdot x}$. We also give a formula for the trace of this operator. Both of these results abstract the previous work of Kirillov which is presented in section 3.3.

We presuppose only basic knowledge of transformation group C^* algebras and representation theory.

All proofs are concluded with Halmos's $\square$. Our numbering system is sequential, which we feel makes the paper easier to read and back-reference when necessary.

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Chapter 1

Preliminaries

Discussion 1

The purpose of this section is to inform the reader of some facts about the representations of transformation group C^* algebras. More detailed accounts may be found in [6], [8], [18] and [25].

Section 1.1

C^* algebras and representations

Discussion 2

Let Ω be a locally compact Hausdorff space. Denote the space of complex valued functions on Ω vanishing at infinity by $C_0(\Omega)$. Denote by $C_c(\Omega)$ the subspace functions in $C_0(\Omega)$ with compact support.

We suppose that G is a locally compact group with a jointly continuous action on Ω, that is, we have a continuous map $G \times \Omega \mapsto \Omega$, where for $r, s \in G$, $r \cdot (s \cdot x) = (rs) \cdot x$, where $s \cdot x$ is the image of (s, x) in Ω. As (G, Ω) is a locally compact transformation group, we may form the transformation group C^* algebra, denoted $C^*(G, \Omega)$. For a precise treatment of the details, see [6], [8], [18], [25] and [26]. We here give the basics. Let μ be a fixed Haar measure on G. First we define the Banach *-algebra $L^1(G, \Omega)$ of all Bochner integrable $C_0(\Omega)$-valued measurable functions on G with respect to μ, with multiplication and involution defined by

$$f * g(s, x) = \int\limits_{r \in G} f(r, x) g(r^{-1}s, r^{-1} \cdot x) \mathrm{d}\mu_G(r)$$

$$f^*(s,x) = \Delta(s^{-1})\overline{f}(s^{-1}, s^{-1}\cdot x)$$

where $f, g \in L^1(G, C_0(\Omega))$, $s \in G$, $x \in \Omega$, Δ is the modular function on G, and $\overline{}$ denotes complex conjugation.

$C^*(G,\Omega)$ is the enveloping C^* algebra of the Banach-* algebra $L^1(G,\Omega)$ under the multiplication and involution as just defined.

When Ω is a single point, the above construction gives the group C^* algebra of G. We will make frequent use of the dense subalgebra $C_c(G\times\Omega)$, which we denote by $C_c(G,\Omega)$.

By Ω/G we mean the orbit space with the quotient topology.

Definition 3

A *covariant representation* $L = (V_L, M_L)$ of (G,Ω) on a Hilbert space H_L consists of a uniformly bounded strongly continuous unitary representation V_L of G on H_L, and a norm-decreasing non-degenerate *-preserving representation of $C_0(\Omega)$, M_L on H_L such that $V_L(s)M_L(\phi)V_L(s^{-1}) = M_L({}^s\phi)$

We will henceforth assume all representations to be non-degenerate. We must establish a lemma about these covariant representations.

Lemma 4. *A covariant representation $L = (V_L, M_L)$ on H_L gives a representation of $L^1(G,\Omega)$, also called L, by*

$$L(f)(\xi) = \int_{s\in G} M_L(f(s,\cdot))V_L(s)(\xi)\, d\mu_G(s).$$

Proof.

See [15], Theorem 1.5. □

Example 5

As an example, the natural representation of $L^1(G,\Omega)$ upon $L^2(G,\Omega)$ by left multiplication is the integrated form of (V, M), where

$$(V(s)f)(r,x) = f(s^{-1}r, s^{-1}\cdot x), \qquad (M(\phi)f)(r,x) = \phi(x)f(r,x),$$

for $r, s \in G,\ x \in \Omega,\ f \in L^1(G, C_0(\Omega))$, and $\phi \in C_0(\Omega)$. From these actions, we induce homomorphisms R_G and R_Ω of G and $C_0(\Omega)$ into the multiplier algebra, $M(C^*(G, \Omega))$, see [2], of $C^*(G, \Omega)$. Phil Green in [18] has combined this with the work in [6] to show the following:

Proposition 6. *If H is a Hilbert space then there is a one to one correspondence between *-representations of $C^*(G, \Omega)$ on H and covariant representations of (G, Ω) on H. The correspondence is given in one direction by Lemma 2 (recall that *-representations of $L^1(G, C_0(\Omega))$ are in 1-1 correspondence *-representations of its enveloping C^* algebra, $C^*(G, \Omega)$). In the other direction, let L be a *-representation of $C^*(G, \Omega)$. Then L has a unique extension to $M(C^*(G, \Omega))$, which we will also denote by L, and*

$$V_L(s) = L(R_G(s)), \qquad\qquad M_L(\phi) = L(R_\Omega(\phi)).$$

Proof.

See [18], page 195. □

We will refer to the notation in Example 4 again later on.

Section 1.2

Topologies of representation and ideal spaces

Discussion 7

The purpose of this section is to describe the topologies on the representation and primitive ideal spaces of C^* algebras, as well as some information about how these topologies behave in the transformation group C^* algebra case.

Definition 8

Let A be a fixed C^* algebra. Define $\widehat{A}$ to be the space of unitary equivalence classes of irreducible representations, and define $\mathrm{Rep}(A)$ to be the space of all of all *-representations of A. Let $T \in \mathrm{Rep}(A)$, and $\mathcal{S} \subseteq \mathrm{Rep}(A)$. We say that T is *weakly contained* in $\mathcal{S}$ (notation $T \prec \mathcal{S}$) if

$$\mathrm{kernel}(T) \supseteq \bigcap_{S \in \mathcal{S}} \mathrm{kernel}(S)$$

or equivalently (see [11]), if every positive functional associated to T may be weakly-* approximated by sums of positive functionals associated to $\mathcal{S}$. We comment that [5], section 3.4 also has a nice discussion of this topology. Restricted to $\widehat{A}$, the notion of weak containment defines the closure operation for the *hull-kernel topology* of $\widehat{A}$, where the closure of a set is the set of representations which are weakly contained in the set. From this, each $T \in \mathrm{Rep}(A)$ is weakly equivalent to a unique closed subset $\mathrm{Sp}(T)$ of $\widehat{A}$, called the *spectrum* of T. For $\mathcal{F}$ a finite family of nonvoid open sets of $\widehat{A}$, let

$$U(\mathcal{F}) = \big\{T \in \mathrm{Rep}(A) \mid \mathrm{Sp}(T) \cap B \neq \emptyset \text{ for each } B \text{ in } \mathcal{F}\big\}.$$

The *inner-hull kernel* topology of $\mathrm{Rep}(A)$ is the topology in which the set of all $U(\mathcal{F})$ (as above) form a basis for the open sets. Restricted to $\widehat{A}$, this is the hull-kernel topology. In this paper, we will always assume that $\mathrm{Rep}(A)$ and $\widehat{A}$ are equipped with the inner-hull-kernel and the hull-kernel topologies, respectively.

Proposition 9. *If $T \in Rep(A)$ and $\mathcal{S} \subseteq Rep(A)$, the following are equivalent:*

(1) T is weakly contained in $\mathcal{S}$

(2) The closure of $\bigcup_{S \in \mathcal{S}} Sp(S)$ contains $Sp(T)$

(3) T is in the closure of the set $\mathcal{S}_f$ of all finite direct sums of elements of $\mathcal{S}$. If T is irreducible, these conditions hold if and only if T belongs to the closure of S.

Proof.

See Theorems 1.1 and 2.3 of [12] and also the remark following Theorem 2.1 of [10]. We comment that this proposition is first stated as such in [13], page 427. □

We also state the following easy facts:

Proposition 10. *A sequence $\{T_n\}_{n=1}^{\infty}$ converges to T in Rep(A) if and only if every subnet weakly contains T.*

Proposition 11. *If $T_n \to T$ in Rep(A) and T weakly contains S, then $T_n \to S$ in Rep(A).*

Definition 12

A *primitive ideal* of a C^* algebra A is the kernel of an irreducible representation of A. The set of all primitive ideals we denote by $\mathrm{Prim}(A)$. We topologize $\mathrm{Prim}(A)$ by defining a closure operation: If $S \subseteq \mathrm{Prim}(A)$, let

$$\ker(S) = \begin{cases} \bigcap_{J \in S} J, & S \neq \emptyset \\ \mathrm{Prim}(A), & S = \emptyset \end{cases},$$

this is a closed ideal (unless $S = \emptyset$) called the *kernel* of S. Now if T is any subset of A, define

$$\mathrm{hull}(T) = \{I \in \mathrm{Prim}(A) \mid I \supseteq T\}.$$

This is called the *hull* of T. Then

$$S \mapsto \overline{S} = \mathrm{hull}(\ker(S))$$

is the closure operation defining the *Jacobson structure topology* on $\mathrm{Prim}(A)$.

Proposition 13. *The above construction defines a topology on Prim*(A)*.*

Proof.

See [24], Theorem 5.4.6. □

Definition 14

We define a C^* algebra A to be *Type I* (read: "type one") if for every non-zero irreducible representation T on A we have $T(A) \supseteq K(H_T)$, where $K(H_T)$ denotes the compact operators on H_T, the Hilbert space of T. Such a C^* algebra is also called *postliminal*, or *generalized completely continuous (G.C.R.)* (The R is for *representations*). We also comment that if $T(A) = K(H_T)$, for every nonzero irreducible representation T on A, we call A *completely continuous (C.C.R)* or *liminal.* A C.C.R. algebra is obviously G.C.R. as well.

The following two theorems give us part of what we will need about Type I C^* algebras for this paper.

Proposition 15. *Let A be a C^* algebra. The following three conditions are equivalent:*

(1) $\widehat{A}$ *is a* T_0 *space.*

(2) Two irreducible representations of A with the same kernel are equivalent.

(3) The canonical map $\widehat{A} \mapsto$ *Prim*(A) *is a homeomorphism.*

Proof.

See [5], Proposition 3.1.6, page 71. □

Now we give a theorem particular to transformation group C^* algebras. We comment that the second hypothesis of the next is trivial in our case as our stability subgroups are connected and simply connected by hypothesis, and connected, simply connected nilpotent Lie groups are Type I.

Theorem 16. *$C^*(G,\Omega)$ is Type I if and only if Ω/G is T_0 and all isotropy subgroups are Type I.*

Proof.

See [16], Theorem 3.3. □

Definition 17

Let G be a group acting on a space Ω, and assume $x \in \Omega$. We will always in this paper denote the isotropy subgroup of x in G by G_x.

We also need to include the following result.

Proposition 18. *Let (G,Ω) be a Polish (i.e., both are separable and metrizable by a complete metric) transformation group. Then following two statements are equivalent.*

(1) For each $x \in \Omega$, the map $sG_x \mapsto s \cdot x$ onto $G \cdot x$ is a homeomorphism.
(2) Ω/G is a T_0 space.

Proof.

See [8], Theorem 2.1. □

Theorem 19. *If I is a proper closed ideal of a C^* algebra A, then*

$$I = ker(hull(I));$$

that is, I is the intersection of the primitive ideals containing it.

Proof.

See [24], Theorem 5.4.3. □

Definition 20

For A a C^* algebra, we here give the space $\mathcal{I}(A)$ of arbitrary closed two sided ideals of A the topology having as a sub-base the sets

$$\{\mathcal{O}_J\}_{J\in\mathcal{I}(A)}, \text{ where } \mathcal{O}_J = \{I \in \mathcal{I}(A) \mid I \not\supset J\}.$$

We may also identify any ideal with a closed subset of Prim(A) by the last theorem. The topology of $\mathcal{I}(A)$ may then be described in terms of a sub-base for the topology on $\mathcal{K}(\widehat{A})$, the closed subsets of $\widehat{A}$ (see section 1.4 below). This sub-base is the family $U(O) = \{F \in \mathcal{K}(\widehat{A}) \mid F \cap O \neq \emptyset\}$ where O runs over the the set of all open subsets of $\widehat{A}$. We comment that on Prim(A) this topology restricts to the usual hull-kernel (Jacobson) topology. One can see that this topology is almost Fell's "inner hull kernel" topology, see [10]; we define that topology in section 1.4.

Proposition 21. *For A a separable C^* algebra, the topology of $\widehat{A}$ has a countable base.*

Proof.

See [5], Proposition 3.3.4. $\square$

The next lemma will be of importance in Chapter 2 in the proof of Lemma 154.

Lemma 22. *Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a net of ideals in $\mathcal{I}(A)$ converging to I. Suppose also that I_α corresponds to $F_\alpha \in \mathcal{K}(\widehat{A})$. Then, given any $P \in F$, there is a subnet, $\{I_\beta\}_{\beta\in\Lambda'}$, such that there are $P_\beta \in F_\beta$ with $\{P_\beta\}_{\beta\in\Lambda'}$ converging to P in Prim(A).*

Proof.

See [26], Lemma 2.4 on page 338. $\square$

Definition 23

Let D' and F be C^* algebras and D be an ideal of D'. Suppose that P is a *-homomorphism from F into D'. Define

(1) $P_* : \mathcal{I}(F) \mapsto \mathcal{I}(D)$ by

$$P^*(J) = \text{the ideal generated by } \{P(f)d \mid f \in J, d \in D\}$$

(2) $P^* : \mathcal{I}(D) \mapsto \mathcal{I}(F)$ by

$$P^*(I) = \{f \in F \mid P(f) \cdot D \subseteq I\}.$$

Note that $P^*(I)$ is an ideal of F as I is an ideal of D'. The properties that we need are given by the next lemma.

Lemma 24.

(1) P^* is continuous from $\mathcal{I}(D) \mapsto \mathcal{I}(F)$.

(2) P^* preserves arbitrary intersections while P_* preserves arbitrary unions (the union of a family of ideals being the ideal that they generate).

Proof.

This is part of [18], Proposition 9(i). □

Definition 25

If $L = (V, M)$ is a *-representation of $C^*(G, \Omega)$, let $\mathrm{Res}_H^G(L)$ denote the *-representation of $C^*(H, \Omega)$ corresponding to $(V|_H, M)$.

Let $R_H = R_G|_H$ and note that $R = (R_H, R_\Omega)$ is a homomorphism of $C^*(H, \Omega)$ into $M(C^*(G, \Omega))$, see Example 4 for notation. As $M(C^*(G, \Omega))$ is a C^* algebras, the integrated form of R "lifts" to a *-homomorphism of $L^1(H, \Omega)$ into $M(C^*(G, \Omega))$. We refer to this homomorphism as $\mathcal{L}$.

Definition 26

We define $\mathrm{Res}_H^G : \mathcal{I}(C^*(G,\Omega)) \mapsto \mathcal{I}(C^*(H,\Omega))$ by $\mathrm{Res}_H^G = \mathcal{L}^*$. When $H = e$, we define $\mathrm{Res} = \mathrm{Res}_e^G : \mathcal{I}(C^*(G,\Omega)) \mapsto \mathcal{I}(C_0(\Omega))$; we will use this notation without reference.

The next lemma justifies the ambiguous notation of the last two definitions.

Lemma 27. *If L is a representation of $C^*(G,\Omega)$, then*

$$ker(Res_H^G(L)) = Res_H^G(ker(L)).$$

Proof.

See [18], Proposition 9(ii). $\square$

Definition 28

Let Γ be the integrated form of the homomorphism R_G of G into $M(C^*(G,\Omega))$.

We need another result relating to kernels.

Lemma 29. *Let A be a C^* algebra with norm $\|\cdot\|$, and L_1 and L_2 be two representations of A with $I = ker(L_1) = ker(L_2)$. Denote the norms on the C^* algebras $L_1(A)$ and $L_2(A)$ by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. For each $a \in A$ we have $\|L_1(a)\|_1 = \|L_2(a)\|_2$.*

Proof.

Note that norms are Banach algebra quotient norms, so

$$\|L_1(a)\|_1 = \|L_2(a)\|_2 = \inf_{b\in I} \left\{\|a+b\|\right\},$$

and the result is clear from here. $\square$

Section 1.3

Nilpotent Lie algebras and groups

Discussion 30

The purpose of this section is to give the information about nilpotent Lie groups that will be needed for this thesis. We will describe their representation spaces in later sections.

Discussion 31

We assume that the reader has a basic knowledge of topological groups and Lie groups.

Throughout this paper, we use the following conventions:

1) All groups are real, locally compact, separable, connected and simply connected.
2) All representations are unitary.

Let G denote a Lie group and $\mathfrak{g}$ its Lie algebra. We assume that $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$. We comment that the convention of referring to Lie groups in uppercase Roman and Lie algebras in lowercase Gothic will be used throughout this paper. We also refer to elements of the group in lower case Roman and elements of the Lie algebra in uppercase Roman, so notation such as "$x \in G$" and "$X \in \mathfrak{g}$" will be common.

Definition 32

The *adjoint representation*, ad of $\mathfrak{g}$ on $\mathfrak{g}$ is defined as $\mathrm{ad}_x : \mathfrak{g} \mapsto gl(\mathfrak{g})$ by $\mathrm{ad}_x(y) = [x, y]$, for all $y \in \mathfrak{g}$ (here $[\cdot, \cdot]$ denotes Lie bracket on $\mathfrak{g}$).

Definition 33

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if ad_x is a nilpotent endomorphism of $\mathfrak{g}$, for all $x \in \mathfrak{g}$.

The Lie group G is *nilpotent* if $\mathfrak{g}$ is nilpotent.

The following two lemmas are immediate from the definitions.

Lemma 34.

1) If $\mathfrak{g}$ is nilpotent, then all quotient algebras and subalgebras of $\mathfrak{g}$ are nilpotent.

2) If G is nilpotent, then all closed connected subgroups of G are nilpotent.

3) If G is nilpotent, and N is a closed connected normal subgroup, then G/N is nilpotent.

Lemma 35. *Let $\mathfrak{g}^0 = \mathfrak{g}$, define $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$, for $i = 0, 1, 2, ...$, then*

1) Each $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$,

2) $\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \; ....$

3) $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}^k = \{0\}$ for some $k > 0$. In this case, the $\mathfrak{g}^i$'s are a strictly decreasing sequence.

Definition 36

The sequence of subalgebras defined in Lemma 35 is called the *descending central series* for $\mathfrak{g}$.

Lemma 37. *Let $\mathfrak{h}$ be a subalgebra of codimension 1 in a nilpotent Lie algebra. Then $\mathfrak{h}$ is an ideal, in fact, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$.*

Proof. See [3], Lemma 1.1.8, page 4. $\square$

Discussion 38

Let G be a connected Lie group, with exponential map $\exp : \mathfrak{g} \mapsto G$, and for X and Y in $\mathfrak{g}$, define a map $* \;:\; \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ by

$$X * Y = \log(\exp(X) \cdot \exp(Y)).$$

This is an analytic function well defined near $X = Y = 0$, and does not depend upon which of the locally isomorphic connected Lie groups G we associate to $\mathfrak{g}$. This is also given by a universal power series which involves only commutators. The resultant formula is know as the *Campbell-Baker-Hausdorff formula*, which lets us construct G locally, only knowing the structure of $\mathfrak{g}$. For our purposes, the formula is given by

$$X * Y = (X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] +$$

$$-\frac{1}{48}[Y,[X,[X,Y]]] - \frac{1}{48}[X,[Y,[X,Y]]] +$$

$$+ \text{ commutators in five or more terms})$$

but the general term can not be expressed in closed form. For further discussion of this, see [3], pp. 11-12.

Lemma 39. *Let G be a (connected, simply connected) nilpotent Lie group, with Lie algebra $\mathfrak{g}$.*

(a) exp is an analytic diffeomorphism.

(b) The Campbell-Baker-Hausdorff formula holds for all $X, Y \in \mathfrak{g}$.

Proof.

See [3], Theorem 1.2.1, page 13. □

Corollary 40. *Every connected Lie subgroup of G is closed and simply connected. Every Lie subgroup H of a (connected, simply connected) nilpotent Lie group is closed and simply connected.*

Proof.

See [3], Corollary 1.2.2, page 14. □

Definition 41

Let V and W be two finite dimensional vector spaces. We say that a map $f:\ V \mapsto W$ is *polynomial* if it is described by polynomials for some (hence any) pair of bases. We say that f is a *polynomial diffeomorphism* if f^{-1} exists and both f and f^{-1} are polynomial.

Theorem 42. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_k \subseteq \mathfrak{g}$ be subalgebras, with dim$(\mathfrak{g}_j) = m_j$ and dim$(\mathfrak{g}) = n$.*

(a) $\mathfrak{g}$ *has a basis $\{X_1, \ldots, X_n\}$ such that*

(1) for each m, $\mathfrak{h}_m = \mathbb{R}-span\{X_1, \ldots, X_m\}$ is a subalgebra of $\mathfrak{g}$,

(2) for $1 \leq j \leq k$, $\mathfrak{h}_{m_j} = \mathfrak{g}_j$.

(b) If the $\mathfrak{g}_j$ are ideals of $\mathfrak{g}$ then one can pick the X_j so that (1) *is replaced by*

(3) for each m, $\mathfrak{h}_m = \mathbb{R} - span\{X_1, \ldots, X_m\}$ is an ideal of $\mathfrak{g}$.

Proof.

See [3], Theorem 1.1.13, page 10. □

Definition 43

We call a basis satisfying (1) and (2) a *weak Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$, and one satisfying (2) and (3) a *strong Malcev basis* for $\mathfrak{g}$ through $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$.

Proposition 44. *Let $\{X_1, \ldots, X_n\}$ be a strong Malcev basis for a nilpotent Lie algebra $\mathfrak{g}$. Define a map $\phi : \mathbb{R}^n \mapsto G$ by*

$$\phi(s) = exp(s_1X_1) \cdots exp(s_nX_n) = exp(s_1X_1 * \cdots * s_nX_n).$$

(1) $\phi(s) = exp\left(\sum_{j=1}^{n} P_j(s)X_j\right)$, where the P_j are polynomial.

(2) $P_j(s) = s_j +$ a polynomial in $s_{j+1}, \ldots, s_n$.

(3) *$\log \circ \phi$ is a polynomial diffeomorphism with polynomial inverse.*

(4) *If $\mathfrak{g}_k = \mathbb{R} - span(X_1, \ldots, X_k)$ and $G_k = exp(\mathfrak{g}_k)$, then*

$$G_k = exp(\mathbb{R}X_1) \cdots exp(\mathbb{R}X_k).$$

Proof.

See [3], Proposition 1.2.7, page 16. □

We have a slightly weaker result for weak Malcev bases.

Proposition 45. *Let $\{X_1, \ldots, X_n\}$ be a weak Malcev basis for a nilpotent Lie algebra $\mathfrak{g}$. Define a map $\psi : \mathbb{R}^n \mapsto G$ by*

$$\psi(s) = exp(s_1X_1) \cdots exp(s_nX_n) = exp(s_1X_1 * \cdots * s_nX_n).$$

(1) *ψ is a polynomial diffeomorphism with polynomial inverse.*

(2) *If $\mathfrak{g}_k = \mathbb{R} - span\{X_1, \ldots, X_k\}$ and $G_k = exp(\mathfrak{g}_k)$, then*

$$G_k = exp(\mathbb{R}X_1) \cdots exp(\mathbb{R}X_k).$$

Proof.

See [3], Proposition 1.2.8, page 17. □

Theorem 46. *Let G be an n-dimensional nilpotent Lie group with Lie algebra $\mathfrak{g}$.*

(1) *The map exp : $\mathfrak{g} \mapsto G$ takes Lebesgue measure on $\mathfrak{g}$ to a left-invariant (Haar) measure on G. This meaure is also right-invariant.*

(2) *Let $\phi : \mathbb{R}^n \mapsto G$ be any polynomial coordinate map. Then ϕ takes Lebesgue measure on $\mathbb{R}^n$ to a Haar measure on G. In particular, this is true if Lebesgue measure is transferred to G via strong or weak Malcev*

coordinates.

Proof.

See [3], Theorem 1.2.10, page 19. □

Weak Malcev coordinates let us choose a nice cross section for G/H, and give us a nice description of the invariant measure, as in the next.

Theorem 47. *Let $\mathfrak{h}$ be a k-dimensional subalgebra of the nilpotent Lie algebra $\mathfrak{g}$, let $H = exp(\mathfrak{h})$ and $G = exp(\mathfrak{g})$, and let $\{X_1, \ldots, X_n\}$ be a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$. Define $\phi : \mathbb{R}^{n-k} \mapsto G/H$ by*

$$\phi(t_1, \ldots, t_{n-k}) = H \cdot exp(t_1 X_{k+1}) \cdots exp(t_{n-k} X_n).$$

Then ϕ is an analytic diffeomorphism taking Lebesgue measure on $\mathbb{R}^{n-k}$ to a G-invariant measure on G/H.

Proof.

See [3], Theorem 1.2.12, page 21. □

We also use the following general result.

Lemma 48. *Let H be a closed subgroup of a locally compact group G. Suppose that G/H has a right invariant measure $d\mu_{G/H}$; denote right Haar measure on H by $d\mu_H$. Then right Haar measure $d\mu_G$ on G is given by*

$$\int_G \phi(g)\, d\mu_G(g) = \int_{\dot{g} \in G/H} \left(\int_{h \in H} \phi(\dot{g}h)\, d\mu_H(h) \right) d\mu_{G/H}(\dot{g}),$$

for all $\phi \in C_c(G)$.

Proof.

See [3], Lemma 1.2.13, page 22. □

Definition 49

Let G be a connected, simply-connected nilpotent Lie group. Define the *adjoint representation* Ad of G into $\mathrm{GL}(\mathfrak{g})$ as follows: for $x, y \in G$, let $\alpha_x(y) = xyx^{-1}$. Its differential at the unit element, $\mathrm{Ad}(x)$, makes the following diagram commute for any $x \in G$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(x)} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\alpha_x} & G. \end{array}$$

It satisfies the formula

$$\exp(\mathrm{Ad}(x)Y) = x(\exp(Y))x^{-1}.$$

We relate Ad and ad by the following:

$$\mathrm{Ad}(\exp(x)) = \exp(\mathrm{ad}_x).$$

That is to say, the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\mathrm{ad}]{} & \mathrm{gl}(\mathfrak{g}). \end{array}$$

For more on this, see [3], page 12.

We comment further that in our setting, that of nilpotent Lie groups, that for $Y \in \mathfrak{g}$, that $\mathrm{Ad}(x)(Y) = xYx^{-1}$, it is easy to see that this is multiplicative, $\mathrm{Ad}(xy) = \mathrm{Ad}(x)\mathrm{Ad}(y)$.

The following four observations are of importance in Chapter 3.

Lemma 50. *Let $\mathfrak{h}$ be a codimension one subalgebra of $\mathfrak{g}$. Let a basis of $\mathfrak{g}$ be $\{X_1, \ldots, X_n\}$ (dim($\mathfrak{g}$) $= n$), with $\mathfrak{h} = \mathbb{R} - span\{X_1, \ldots, X_{n-1}\}$. Let an arbitrary element $S \in \mathfrak{h}$ be $S = t_1X_1 + \cdots + t_{n-1}X_{n-1}$, and $R \in \mathfrak{g}/\mathfrak{h}$ as t_nX_n, then if we write $s = exp(S)$, $r = exp(R)$, we have*

$$rsr^{-1} = exp\left(\sum_{j=1}^{n-1}\left(t_jX_j + t_j \cdot \sum_{\substack{k=1\\k\neq j}}^{n-1} P_{j,k}(t_n)X_k\right)\right) =$$

$$= exp\left(\sum_{i=1}^{n-1}\left(t_i + \sum_{\substack{j=1\\j\neq i}}^{n-1} t_jP_{j,i}(t_n)\right)X_i\right),$$

where each $P_{j,k}$ is a polynomial in t_n.

Proof.

Remember that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, see Lemma 37. We have

$$rsr^{-1} = \exp(t_nX_n)\exp(t_1 + \ldots t_{n-1}X_{n-1})\exp(-t_nX_n) =$$

$$\exp\big(\exp(t_nX_n)(t_1X_1 + \cdots + t_{n-1}X_{n-1})\exp(-t_nX_n)\big) =$$

$$\exp\left(e^{\mathrm{ad}(t_nX_n)}\left(\sum_{j=1}^{n-1} t_jX_j\right)\right) = \exp\left(\left(\sum_{j=1}^{n-1} e^{\mathrm{ad}(t_nX_n)}(t_jX_j)\right)\right) =$$

$$\exp\left(\sum_{j=1}^{n-1}\sum_{k=1}^{\infty}\frac{\big(\mathrm{ad}(t_nX_n)\big)^k(t_jX_j)}{k!}\right) =$$

$$\exp\left(\sum_{j=1}^{n-1}\left(t_jX_j + [t_nX_n, t_jX_j] + \frac{[t_nX_n,[t_n, t_jX_j]]}{2!} + \ldots\right)\right) =$$

$$\exp\left(\sum_{j=1}^{n-1}\left(t_jX_j + t_nt_j[X_n, X_j] + \frac{t_n^2t_j}{2!}[X_n,[X_n, X_j]] + \ldots\right)\right).$$

Note that by nilpotence, in the j_{th} summand none of the brackets can "return" on X_j; the sum is finite. So the last gives us

$$rsr^{-1} = \exp\left(\sum_{j=1}^{n-1}\left(t_jX_j + t_j \cdot \sum_{\substack{k=1\\k\neq j}}^{n-1} P_{j,k}(t_n)X_k\right)\right),$$

for some collection of polynomials $\{P_{j,k}\}^{n-1}_{\substack{j,k=1\\k\neq j}}$ in t_n, what we set out to prove, and the second formula is a simple rearrangement. □

Corollary 51. *In the setting of Lemma 50, we have H normal in G.*

Lemma 52. *Let H be a normal subgroup of G. Denote Haar measure on H as μ, then for any $x \in G$, and measurable set E of H, we have $\mu(xEx^{-1}) = \mu(E)$.*

Proof.

This is part of Proposition 1.2.9, page 17 of [3]. □

Lemma 53. *In the setting of connected, simply connected nilpotent Lie groups, we have $\mu(E) = \mu(E^{-1})$, where $E^{-1} = \{e^{-1} \mid e \in E\}$, for Haar measure μ.*

Proof.

Haar measure on G is lifted from a Lebesgue measure on $\mathfrak{g}$, see Theorems 45 and 46, and as for any $x \in G$ such that $x = \exp(X)$, $X \in \mathfrak{g}$, we have $x^{-1} = \exp(-X)$, the result is clear as $X \mapsto -X$ is an orthogonal linear transformation; hence it preserves measure on $\mathbb{R}^n$ ($\dim(\mathfrak{g}) = n$). □

Section 1.4

A topology on the space of closed subsets of a locally compact space

Definition 54

Let X be an arbitrary locally compact space. Let $\mathcal{K}(X)$ denote the set of all closed subsets of X, and equip $\mathcal{K}(X)$ with the topology whose open sets have as basic open neighborhoods $U(C,\mathcal{F}) = \{F \in \mathcal{K}(X) \mid F \cap C = \emptyset,\ F \cap O \neq \emptyset \text{ for all } O \in \mathcal{F}\}$ where C is a compact subset of X and $\mathcal{F}$ is a finite family of open sets of X. This topology is call the *compact-open topology* of $\mathcal{K}(X)$.

Lemma 55. *$\mathcal{K}(X)$ equipped with the compact-open topology is a compact Hausdorff space.*

Proof.

See [10], Theorem 1. □

Lemma 56. *If Y is a subset of X, then the compact-open topology of $\mathcal{K}(Y)$ is the compact-open topology of $\mathcal{K}(X)$ relativized to $\mathcal{K}(Y)$.*

Proof.

The proof follows from the fact that every compact (resp. open) subset of Y is the intersection of Y with with a compact (resp. open) subset of X. □

Lemma 57. *If X has a countable base of open sets, then $\mathcal{K}(X)$ also has a countable base of open sets.*

Proof.

Let $\{O_n\}_{n=1}^{\infty}$ be a basis of open sets of X such that the closure of each O_n is compact. Define $C_n = \overline{O_n}$. Then the sets $\{U(C_n, \mathcal{F})\}$, where $\mathcal{F}$ is a finite subcollection of $\{O_n\}$ form a basis of $\mathcal{K}(X)$. □

We henceforth assume that X is locally compact Hausdorff and separable.

Lemma 58. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{K}(X)$ converging to $F \in \mathcal{K}(X)$. Assume $x_n \in F_n$ for all n and that $x_n \to x$ in X. Then $x \in F$.*

Proof.

Suppose that $x \notin F$. As a locally compact Hausdorff space is regular, separability here implies metrizability (see [20], page 146 and Theorem 16, page 125), hence X is T_4. So there exists a compact neighborhood C' of x such that $F \cap C' = \emptyset$. But for sufficiently large n, we have $x_n \in C'$, so $F_n \cap C' \neq \emptyset$. Let $U(C, \mathcal{F})$ be a basic open neighborhood of F, and consider $U(C \cup C', \mathcal{F})$. This is a basic open neighborhood containing F but for sufficiently large n, not containing F_n. This is a contradiction to the fact that $F_n \to F$.

Lemma 58. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{K}(X)$ converging to F. For each $x \in F$, there exists elements $x_n \in F_n$ with $x_n \to x$ in X.*

Proof.

Let $x \in F$, and let O be an arbitrary open neighborhood of x in X. Let $U(C, \mathcal{F})$ be a basic open neighborhood in $\mathcal{K}(X)$ containing F. Defining $\mathcal{F}' = \mathcal{F} \cup O$, then $U(C, \mathcal{F}')$ is also a neighborhhood of F, and so for sufficiently large n, $F_n \in U(C, \mathcal{F}')$, that is, $F_n \cap O \neq \emptyset$, that is, $\{F_n\}_{n=1}^{\infty}$ eventually intersects O. Combined with the fact that X is second countable implies the desired result. $\square$

Lemma 59. *Let X be Hausdorff, and $\{F_n\}_{n=1}^{\infty}$ a sequence in $\mathcal{K}(X)$. If the following two conditions hold:*

1) given $x \in F$, there exists $x_n \in F_n$ such that $x_n \to x$ in X,

2) if $x_n \in F_n$ and $x_n \to x$ in X, then $x \in F$,

then

$F_n \to F$ in $\mathcal{K}(X)$.

Proof.

Let $U(C, \mathcal{F})$ be a basic open neighborhood of F. For each $O \in \mathcal{F}$, let $x \in F \cap O$. By 1), there exists an integer N_O, such that when $n \geq N_O$, $F_n \cap O \neq \emptyset$. By the finiteness of the collection O, we can choose one integer N that works for all $O \in \mathcal{F}$.

Now we observe that if infinitely many of the F_n intersect C, we could choose a subsequence $\{y_{n_j}\}_{j=1}^{\infty}$ with $y_{n_j} \to y$ in X, where $y_{n_j} \in F_{n_j} \cap C$, where $\{F_{n_j}\}_{j=1}^{\infty}$ is a subsequence of $\{F_n\}_{n=1}^{\infty}$. The limit of this sequence must be contained in $F \cap C$ (we observe that C is closed as X is Hausdorff). This contradicts the fact that $F \cap C = \emptyset$. So there must exist an integer N' such that when $n \geq N'$, $F_n \cap C \neq \emptyset$.

So when $n \geq \max(N, N')$, we have $F_n \in U(C, \mathcal{F})$. As $U(C, \mathcal{F})$ was arbitrary, we have $F_n \to F$ in $\mathcal{K}(X)$. $\square$

We now let G be a topological group, with Hausdorff topology, and let $\mathcal{K}(G)$ be the set of all closed subgroups of G equipped with the relativized compact-open topology.

Proposition 60. *Let $\{H_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ be two sequences in $\mathcal{K}(G)$ converging to H and K respectively. If $K_n \subseteq H_n$ for all n, then $K \subseteq H$.*

Proof.

This follows immediately from Lemma 58 and Lemma 59. $\square$

Proposition 61. *Let $\{H_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{K}(G)$. Let $H \in \mathcal{K}(G)$. Then $H_n \to H$ if and only if:*

(1) if $h \in G$ and there exists a sequence $\{h_n\}_{n=1}^{\infty}$ such that $h_n \in H_n$ for each n, and $h_n \to h$, then $h \in H$,

(2) for each $h \in H$ there exists a sequence $\{h_n\}_{n=1}^{\infty}$ in G such that $h_n \in H_n$ for each n and $h_n \to h$ in G.

Proof.

This is immediate from Lemmas 58, 59 and 60. □

Lemma 62. *Let G be a simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\{H_n\}_{n=1}^{\infty}$ be a a sequence in $\mathcal{K}(G)$, and let $H \in \mathcal{K}(G)$.Then $H_n \to H$ in $\mathcal{K}(G)$ if and only if*

1) if $\{X_n\}_{n=1}^{\infty}$ is a sequence in $\mathfrak{g}$ converging to $X \in \mathfrak{g}$, and for each n, $X_n \in \mathfrak{h}_n$, then $X \in \mathfrak{h}$,

2) if $X \in \mathfrak{h}$, then there exists a sequence $\{X_n\}_{n=1}^{\infty}$ in $\mathfrak{g}$ such that $X_n \in \mathfrak{h}_n$ for all n and $X_n \to X$ in $\mathfrak{g}$.

Proof.

This follows immediately from Proposition 60 and the fact that exp (resp. ln) are analytic diffeomorphisms of $\mathfrak{g}$ and G (resp. G and $\mathfrak{g}$), see Lemma 45. □

Lemma 63. *Let $\{V_m\}_{m=1}^{\infty}$ be a sequences of subspaces of the n-dimensional (n finite) normed linear space V. Then there exists a sequence of bases $\{x_1^m, x_2^m, ..., x_n^m\}_{n=1}^{\infty}$ of V such that*

1) $x_1^m, ..., x_{k_m}^m$ is a basis of V_m, where dim(V_m) $= k_m$, $m = 1, 2, ...$

2) for all i, $1 \le i \le n$, there exists a subsequence of the sequence $\{x_i^m\}_{m=1}^{\infty}$ that converges in V to some element $x_i \in V$.

3) the elements $x_1, ..., x_n$ are a basis of V.

Proof.

For each m, choose a basis of V_m and expand this basis to a basis of V. We may assume that this basis is orthonormal with respect to an appropriate inner product.

Now as a bounded sequence in a finite dimensional space always has a convergent subsequence, 1) and 2) are satisfied.

By further restricting to subsequences, we may assume that all of $\{x_i^m\}_{m=1}^{\infty}$, $1 \leq i \leq m$ converge in V.

Now it is easy to see that each x_i has norm 1 and the elements $\{x_1, ...x_n\}$ are pairwise orthogonal with respect to the previously defined inner product, thus form a basis of V. $\square$

Lemma 64. *Let G be a nilpotent Lie group, and let $H_m \to H$ in $\mathcal{K}(G)$. If dim$(H_m) = k$ for all n, then dim$(H) = k$.*

Proof.

By Lemmas 62 and 63, the finite dimensional normed linear space $\mathfrak{g}$ posesses bases $x_1^n, ..., x_n^m$ and $x_1, ..., x_n$ such that $x_i^m \to x_i$, $1 \leq i \leq n$. We may assume that $\mathfrak{h}_m$ is spanned by $x_1^m, ..., x_k^m$. Now by Lemma 58, we may assume that $x_1, ..., x_k \in \mathfrak{h}$, and if $x_j \in \mathfrak{h}$ for some $j > k$, then there would exist a sequence $\{X_m\}_{m=1}^{\infty}$ with $X_m \in \mathfrak{h}_m$ for each m, and $X_m \to x_j$ in $\mathfrak{g}$. If this were true, x_j would be a linear combination of $x_1, ..., x_k$, a contradiction. So $x_j \notin \mathfrak{h}$, and $\dim(\mathfrak{h}) = k$, and by the exponential map, $\dim(H) = k$. $\square$

Lemma 65. *Assume that Ω is a locally compact Hausdorff space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in Ω that converges to a point x in Ω. Possibly by passing to a subsequence, we may assume that in $\mathcal{K}(G)$ that the sequence of stabilizers $\{G_{x_n}\}_{n=1}^{\infty}$ converges to a subgroup $S \subseteq G_x$, with strict containment a possibility.*

Proof.

See [9], page 66. □

Definition 66

Suppose that f_0 is a nonnegative, real valued function in $C_c(G)$ that does not vanish at the identity element. For the remainder of this paper, let μ_H be the left Haar measure on H defined by

$$\int_H f_0(t)\mathrm{d}\mu_H(t) = 1$$

such a choice is referred to as a continuous ("smooth") choice of Haar measures, and has the property that $H \mapsto \int_H f\mathrm{d}\mu_H$ is continuous on $\mathcal{K}(G)$ for each $f \in C_c(G)$, see [15], page 908.

Lemma 67. *Suppose that* $\{f_n\}_{n=1}^{\infty} \subseteq C_c(G)$ *converges to* $f \in C_c(G)$ *in the inductive limit topology and* $H_n \to H$ *in* $\mathcal{K}(G)$. *Then*

$$\int_{H_n} f_n d\mu_{H_n} \longrightarrow \int_H f d\mu_H .$$

Proof.

Follows by an $\frac{\epsilon}{2}$ argument. □

Section 1.5

Unitary representations of G and induced representations.

Definition 68

Let G be an arbitrary group. A *unitary representation* T of G on a separable Hilbert space H_T (we will always deal with separable Hilbert spaces in this paper) is a homomorphism $x \mapsto T_x$ of G into the space of all unitary operators on H_T such that if $\xi_1, \xi_2 \in H_T$, the map $x \mapsto \langle T_x(\xi_1), \xi_2 \rangle$ is continuous.

Note that we have $T_{xy} = T_x T_y$ and $T_{x^{-1}} = (T_x)^{-1} = (T_x)^*$. From here on out, the word "representation" will always mean "unitary representation".

Definition 69

Let T_1 and T_2 be two representations of G. T_1 and T_2 are said to be *unitarily equivalent* if there exists a unitary operator $U : H_{T_1} \mapsto H_{T_2}$ such that $(T_2)_x = U(T_1)_x U^{-1}$ for all $x \in G$. In this case, we write $T_1 \cong T_2$.

Let T be a representation of G. A closed subspace N of H_T is called an *invariant subspace* of H_T provided that $T_x(N) \subseteq N$ for all $x \in G$.

T is said to be *irreducible* when H_T has no invariant subspaces under its action except H_T and $\{0\}$.

Definition 70

$\widehat{G}$ is defined to be the set of all equivalence classes of irreducible representations of G.

We now introduce the concept of "induced representation".

Definition 71

Let N be a closed subgroup of G. A mapping $\phi : G/N \mapsto G$ is called a *cross section* if

1) if for each $\dot{g} \in G/N$, $\phi(\dot{g})$ is a coset representative of $\dot{g}$.

2) ϕ is a Borel isomorphism of G/N and $\phi(G/N)$.

3) if C is compact in G/N, then $\phi(C)$ has compact closure in G.

Lemma 72. *If N is a closed subgroup of G, then a cross section of G/N into G exists.*

Proof.

See Lemma 1.1 of [22]. and recall that our groups are separable. □

Definition 73

Assume that we have a Borel measure μ on G/N. For any $x \in G$, define μ_x to be the measure $\mu_x(E) = \mu(E \cdot x)$ for Borel sets $E \subseteq G/N$. The measure μ is called *quasi-invariant* if for all $x \in G$, μ and μ_x are equivalent as measures. Up to equivalence, there is only one quasi-invariant measure. We note that for nilpotent groups that such always exist. For each $x \in G$, we define the Radon-Nikodym derivitive $\rho_x = \frac{\mathrm{d}\mu_x}{\mathrm{d}\mu}$. Now to define the induced representation from a closed subgroup N of G.

Definition 74

Let T be a representation of N. Let μ be a quasi-invariant Borel measure on G/N, let K be the set of all functions $f : G \mapsto H_T$ such that

1) $\langle f(x), \xi \rangle$ is a Borel function of x for all $\xi \in H_T$.

2) $f(xn) = T(n)^{-1} f(x)$ for all $n \in N$ and $x \in G$ (this condition is called covariance).

3) Notably, for $f, g \in K$, the function $x \mapsto \langle f(x), g(x) \rangle$ is always constant on right N cosets, we also require

$$\int_{\dot{s} \in G/N} \|f(\dot{s})\|^2_{H_T} \mathrm{d}\mu(\dot{s}) < \infty.$$

K is a separable Hilbert space with inner product

$$\langle f, g \rangle = \int_{\dot{s} \in G/N} \langle f(\dot{s}), g(\dot{s}) \rangle \mathrm{d}\mu(\dot{s}).$$

For each x and y in G, $f \in K$, define the induced representation U, which we sometimes denote by $\mathrm{ind}_N^G(T)$ by

$$(U_x f)(y) = f(x^{-1}y)\sqrt{\rho_x(y)}.$$

We call U the representation of G induced by the action of T on N. We note that in our case, that of nilpotent Lie groups, that ρ is the constant one; we will henceforth ignore it. We note that the representation U does not depend upon the measure μ. We also note that we will always use left actions in this paper.

Lemma 75. *Let H_1 and H_2 be subgroups of G with $H_1 \subseteq H_2$, let T be a representation of H_1. Then*

$$ind_{H_1}^{G}(T) \cong ind_{H_2}^{G}(ind_{H_1}^{H_2}(T)).$$

Proof.

See Theorem 4.1 of [22]. □

The last lemma is usually referred to as the "inducing in stages" lemma.

Lemma 76. *If $ind_H^G(T)$ is irreducible then T is irreducible on H.*

Proof. See Theorem 10.1 of [22]. □

Section 1.6

The irreducible representations of a connected, simply-connected nilpotent Lie group.

Discussion 77

In this section we describe the irreducible representations of a connected, simply connected nilpotent Lie group and parameterize the representation space of such a group in terms of coadjoint orbits. We will describe the topology on this space in the next section.

Definition 78

Let G be a connected, simply-connected nilpotent Lie group. Henceforth in this paper, "nilpotent Lie group" will mean "connected, simply connected nilpotent Lie group". Denote the dual space of $\mathfrak{g}$ by $\mathfrak{g}^*$. G acts on $\mathfrak{g}^*$ by the *coadjoint map*, Ad^*, precisely, for $x \in G$ and $l \in \mathfrak{g}^*$

$$(\mathrm{Ad}^*(x)l)(Y) = l(\mathrm{Ad}(x^{-1})Y) = l(x^{-1}Yx),\ Y \in \mathfrak{g},\ l \in \mathfrak{g}^*,\ x \in G.$$

The differential $\mathrm{d}(\mathrm{Ad}^*)_e$ of the coadjoint map at the unit $e \in G$ is written $\mathrm{ad}^* : \mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g}^*)$, by

$$(\mathrm{ad}^*(X)l)(Y) = l([X,Y]) = l(\mathrm{ad}(-X)Y),\ X,Y \in \mathfrak{g},\ l \in \mathfrak{g}^*,$$

for more on this, see [3], pages 25-26. We comment that Ad^* is multiplicative; see definition 49 for notation.

The *stabilizer* subgroup of G associated to l is

$$R_l = \{x \in G \mid \mathrm{Ad}^*(x)l = l\},$$

this is related to the Lie subalgebra

$$\mathfrak{r}_l = \{X \in \mathfrak{g} \mid \mathrm{ad}^*(X)l = 0\}$$

by the next lemma.

Lemma 79. *If G is a nilpotent Lie group and $l \in \mathfrak{g}^*$, The stabilizer R_l is connected and $\mathfrak{r}_l$ is its Lie algebra. In particular, $R_l = exp(\mathfrak{r}_l)$.*

Proof.

See [3], Lemma 1.3.1, page 26 □

Lemma 80. *The Ad* orbit of a functional l, $\mathcal{O}_l$, satisfies $\mathcal{O}_l \cong G/R_l$.*

Proof.

This is obvious, see also the end of the proof of Lemma 1.3.1 of [3], page 26. □

Comment 81

Each $l \in \mathfrak{g}^*$ defines a natural bilinear form, $B_l : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{R}$,

$$B_l(X, Y) = l([X, Y]),\ \ X, Y \in \mathfrak{g}.$$

By definition, the radical of B_l is

$$\{Y \in \mathfrak{g} \mid B_l(X, Y) = 0 \text{ for all } X \in \mathfrak{g}\} = \{Y \in \mathfrak{g} \mid l([X, Y]) = 0 \text{ for all } X \in \mathfrak{g}\}$$

$$= \{Y \in \mathfrak{g} \mid (\mathrm{Ad}^*(Y)l) = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Lemma 82. *If $\mathfrak{g}$ is a Lie algebra and $l \in \mathfrak{g}^*$, its radical has even codimension in $\mathfrak{g}$. Hence the coadjoint orbits are of even dimension.*

Proof.

See [3], Lemma 1.3.2 on page 27. □

Definition 83

If V is a real vector space, with a skew-symmetric (symplectic) bilinear form B, its *isotropic* subspaces W with respect to B are those for which $B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. It is known from linear algebra that maximal isotropic subspaces exist and have the same dimension:

$$\frac{1}{2}\dim(V/\mathrm{rad}(B)) + \dim(\mathrm{rad}(B)) = \frac{1}{2}(\dim(V) + \dim(\mathrm{rad}(B)))$$

where $\mathrm{rad}(B) = \{x \in V \mid B(x, y) = 0,\ \text{all } y \in V\}$. ¿From this, they have codimension $k = \frac{1}{2}\dim(V/\mathrm{rad}(B))$; they lie halfway between the radical rad(B) and V. They always include rad(B); for further discussion of this see [3], pp. 27-28 (from which this is abstracted).

Discussion 84

In the representation theory of a nilpotent Lie group G it is important that there exists *subalgebras* $\mathfrak{m} \subseteq \mathfrak{g}$ which are isotropic for B_l and have maximal dimension $n-k$ in $\mathfrak{g}$ (see Theorem 86 below). Such subalgebras are called *polarizing subalgebras* or *maximal subordinate subalgebras* for l. Isotropy insures that $l([\mathfrak{m},\mathfrak{m}]) = 0$. By Discussion 38

$$\exp(M_1)\cdot\exp(M_2) = \\ \exp\left(M_1 + M_2 + \frac{1}{2}[M_1, M_2] + \frac{1}{12}[M_1,[M_1,M_2]] + \text{ higher order terms}\right)$$

and as $M_1, M_2 \in \mathfrak{m}$ (again see Discussion 38), we have that $\chi(\exp(H)) = e^{i\cdot l(H)}$ is a one dimensional representation of the subgroup $M = \exp(\mathfrak{m})$.

We state part of this as a corollary.

Corollary 85. *For $\mathfrak{m}$ an isotropic subalgebra of a functional l on a nilpotent Lie algebra $\mathfrak{g}$, we may define a 1-dimensional representation $\chi_{l,M}$ of $M =$ exp$(\mathfrak{m})$ by $\chi_{l,M}(x) = e^{i\cdot l(\log(x))}$.*

Theorem 86. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let l be any element of $\mathfrak{g}^*$. Then there exists a polarizing subalgebra for l.*

Proof.

See [3], Theorem 1.3.3. □

Definition 87

Now let $l \in \mathfrak{g}^*$, let B_l be the bilinear form $B_l(X,Y) = l([X,Y])$. Choose a maximal subordinate subalgebra $\mathfrak{m}$ for l and let $M = \exp(\mathfrak{m})$. As we have previously noted in Corollary 85, the character

$$\chi_{l,M}(\exp(Y)) = e^{i\cdot l(Y)}, \ Y \in \mathfrak{m}$$

is a one dimensional representation of M. We now define the induced representation $\pi_{l,M}$ of the group G to be $\pi_{l,M} = \operatorname{ind}_M^G(\chi_{l,M})$.

The following results describe the irreducible representation space $\widehat{G}$ in terms of these induced representations.

Theorem 88. *Let $l \in \mathfrak{g}^*$. Then there exists a maximal subordinate subalgebra $\mathfrak{m}$ for l such that $\pi_{l,M}$ is irreducible.*

Proof.

See [3], Theorem 2.2.1. □

Theorem 89. *Let $l \in \mathfrak{g}^*$, and let $\mathfrak{m}, \mathfrak{m}'$ be two maximal subordinate subalgebras for l. Then $\pi_{l,M} \cong \pi_{l,M'}$. (Hence we may write π_l for $\pi_{l,M}$ if we are interested only in equivalence classes of unitary representations.) In particular, $\pi_{l,M}$ is irreducible whenever $\mathfrak{m}$ is maximal subordinate for l.*

Proof.

See [3], Theorem 2.2.2. □

Theorem 90. *Let π be any irreducible unitary representation of G. Then there is an $l \in \mathfrak{g}^*$ such that $\pi_l \cong \pi$.*

Proof.

See [3], Theorem 2.2.3. □

Theorem 91. *Let $l, l' \in \mathfrak{g}^*$. Then $\pi_l \cong \pi_{l'} \iff l$ and l' are in the same $Ad^*(G)$ orbit in $\mathfrak{g}^*$.*

Proof.

See [3], Theorem 2.2.4. □

Section 1.7

The topology on $\widehat{G}$.

Discussion 92

In this section we explicitely describe the Fell topology on the irreducible representation space of a nilpotent Lie group.

Definition 93

Given a locally compact group G, function $f : G \to \mathbb{C}$ is *positive definite* if it is continous and

$$\sum_{i,j=1}^{m} \overline{\lambda_i}\lambda_j f(x_i^{-1}x_j) \geq 0$$

for all finite sets $\{\lambda_1, ..., \lambda_m\} \subseteq \mathbb{C}$ and $\{x_1, ..., x_m\} \subseteq G$. We use the correspondences that exist between

(1) The set $P(G)$ of all positive definite functions on G.

(2) The set of norm bounded linear functionals ω that are positive on the convolution algebra $L^1(G)$:

$$\omega(\phi^* * \phi) \geq 0 \text{ for all } \phi \in L^1(G)$$

(3) The set of all matrix elements $f_{\pi,\xi}(g) = \langle \pi_g \xi, \xi \rangle$, π a unitary representation of G, ξ a vector in H_π.

Definition 94

The following sets from a base for a topology on $\widehat{G}$, the *Fell topology:*

$$U(\pi, \xi_1, ..., \xi_m, K, \epsilon) =$$

$$\{\pi' \in \widehat{G} \mid \exists\, \xi_1', ..., \xi_m' \in H_{\pi'} \text{ such that } |f_{\pi,\xi_i} - f_{\pi',\xi_i'}| < \epsilon \text{ on } K\}.$$

Here $f_\pi, \xi(g) = \langle \pi(g)\xi, \xi \rangle$ as above, $\xi_1, \ldots, \xi_m \in H_\pi$, K is compact in G, and $\epsilon > 0$. For further discussion of this topology, see [4].

Definition 95

If $\mathfrak{S}$ is a set of unitary representations of G, then a representation T is *weakly contained* in $\mathfrak{S}$ if every matrix element $f_{T,\xi}(G) = \langle T_g \xi, \xi \rangle$, $\xi \in H_T$, may be approximated uniformly on compacta by finite sums of matrix elements $f_{S,\eta} = \langle S_g \eta, \eta \rangle$ with $S \in \mathfrak{S}$, $\eta \in H_S$. We write $T \prec \mathfrak{S}$ when this happens, and write $\mathfrak{S} \prec \mathfrak{T}$ for two sets of representations if $S \prec \mathfrak{T}$ for all $S \in \mathfrak{S}$. Also see Definition 6 for how this works on the level of C^* algebras.

Comment 96

As unitarily equivalent representations have the same matrix elements, the above definitions apply to unitary equivalence classes; we have slightly abused notation by referring to concrete representations and their Hilbert spaces.

Lemma 97. *Let G be a locally compact group.*

(a) The sets $U(\pi, \xi_1, ..., \xi_m, K, \epsilon)$ of Definition 94 form a topology basis of $\widehat{G}$.

(b) The smaller family of sets $U(\pi, \xi, K, \epsilon)$ is also a base for this topology.

If G is separable every point $\pi \in \widehat{G}$ has a countable base of neighborhoods.

Proof.

See [4], Lemma N.1.10. □

Proposition 98. *Let G be a locally compact group, $\pi \in G$, and $\mathfrak{S} \subseteq \widehat{G}$. Then $\pi \prec \mathfrak{S} \iff \pi \in Cl(\mathfrak{S})$, the closure in the Fell topology.*

Proof.

See [4], Proposition N.1.13. □

Theorem 99. *If G is a simply connected nilpotent Lie group the Kirillov map $j : \mathfrak{g}^*/Ad^*(G) \mapsto \widehat{G}$ is a homeomorphism.*

Proof.

See [4], Theorem N.3.5. □

Comment 100

By the Kirillov map (Theorem 99), the irreducible representations of G are in 1-1 correspondence with Ad^* orbits in $\mathfrak{g}^*$ and the quotient topology of $\mathfrak{g}^*/G$ describes the topology of $\widehat{G}$. We further comment that Kirillov proved that the above map is continuous, 1-1 and onto, with the bijectivity first proven by Brown in [1] and later given a nicer proof by Joy in [19]. We also comment that for a connected, simply connected nilpotent Lie group that the topology on $\widehat{G}$ is T_1.

Section 1.8

The Inner-Hull-Kernel Topology in Rep(G)

Discussion 101

In this section we describe the topology on arbitrary representations of groups; our focus as usual is nilpotent Lie groups. We also give several results relating to weak containment that will be needed later.

Discussion 102

The inner hull kernel (*ihk*) topology on the space Rep(G) of arbitrary representations of G was introduced by Fell in [12]. As we are restricting our attention to separable groups, it will suffice to consider the set Rep(G) of unitary equivalence classes of representations on separable Hilbert spaces. Rep(G) then includes all cyclic or irreducible representations and is closed under countable direct sums, conjugation, and tensor products.

Definition 103

If $T \in \text{Rep}(G)$ its *spectrum* is $\text{sp}(T) = \{S \in \widehat{G} \mid S \prec T\}$.

The following definition is due to Fell, see [12], page 245.

Definition 104

Let $U_1, ..., U_m$ be open subsets of $\widehat{G}$. The sets

$$N(U_1, ..., U_m) = \{T \in \text{Rep}(G) \mid \text{sp}(T) \cap U_i \neq \emptyset,\ 1 \leq i \leq m\}$$

form a topology, called the *inner-hull-kernel (ihk)* topology on Rep(G).

Equivalently, we can define this in terms of positive definite functions, see [12], Theorem 2.2 on pages 245-248.

Theorem 105. *If $T \in Rep(G)$, let $\xi_1, ..., \xi_m \in H_T$, a compact set $K \subseteq G$, and $\epsilon > 0$ be given. Define $N = N(T, \xi_1, ..., \xi_m, K, \epsilon)$:*

$$T' \in N \iff \exists\ \sigma_i \in sum\{f_{T',\eta} \mid \eta \in H_{T'}\} \text{ such that}$$

$$|f_{T,\xi_i} - \sigma_i| < \epsilon \text{ on } K,\ 1 \leq i \leq m.$$

These sets form a basis for T in the (ihk) topology.

Proposition 106. *If G is a separable locally compact group, the Fell and inner hull kernel topologies coincide on $\widehat{G}$.*

Proof.

See [4], Proposition N.1.18. □

We will need a few more results which are specific to nilpotent Lie groups.

Definition 107

Let G be a nilpotent Lie group, S be a closed connected subgroup, having Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$, repectively. We define $\mathfrak{s}^{\perp}$ to be the set of linear functionals in $\mathfrak{g}^*$ that are zero on $\mathfrak{s}$.

Lemma 108. *Let G be a simply connected nilpotent Lie group, S a closed connected subgroup, and suppose $f \in \mathfrak{g}^*$ satisfies $f([\mathfrak{s}, \mathfrak{s}]) = 0$ so that $\chi_f(exp(Y)) = e^{i \cdot f(Y)}$ is a one dimensional representation of S. Then $W = ind_S^G(\chi_f)$ weakly contains*

$$\{\pi_{f'} \in \widehat{G} \mid f' \in f + \mathfrak{s}^{\perp}\},$$

in fact,

$$Sp(W) = \textit{Fell-closure}\big(\{\pi_{f'} \in \widehat{G} \mid f' \in f + \mathfrak{s}^{\perp}\}\big).$$

Proof.

See [4], Theorem N.2.5. □

Proposition 109. *Let G be a nilpotent Lie group acting on $\mathfrak{g}^*$ by the coadjoint action; let $f \in \mathfrak{g}^*$, and let $\mathfrak{h}$ be a polarizing subalgebra for f, $H = exp(\mathfrak{h})$. Then $Ad^*(H)f = f + \mathfrak{h}^{\perp}$; that is, through each point in a coadjoint orbit there is a 'flat' of dimension half of the dimension of the orbit.*

Proof.

See [3], Proposition 3.1.18, page 97. □

Section 1.9

The topology on subgroup-representation pairs

Discussion 110

In this section we develop a "natural" topology for the set $\mathcal{Q}(G) = \{\langle H, T\rangle \bigm| H \in \mathcal{K}(G)\ ,\ T \in \mathrm{Rep}(G)\}$.

Definition 111

Let G be a locally compact group, $\mathcal{K}(G)$ be the space of closed subgroups of G.

We again define $\mathrm{Rep}(G)$ to be the set of all equivalence classes representations of G.

We define $\mathcal{Y}(G) = \{(K, x) \mid K \in \mathcal{K}(G), x \in K\}$. Using our smooth choice of Haar measures on $\mathcal{K}(G)$, we have:

Lemma 112. *If $f \in C_0(Y)$, the function*

$$K \mapsto \int_K f(K, x) d\mu_K(x)$$

is continuous on $\mathcal{K}(G)$.

Proof.

See Lemma 1.1 of [13]. □

Definition 113

Let Δ_K be the modular function of the closed subgroup K.

Lemma 114. *The map $(K, x) \mapsto \Delta_K(x)$ is continuous on $\mathcal{Y}(G)$.*

Proof.

See the paragraph following Lemma 1.1 of [13]. □

Discussion 115

We make $C_0(\mathcal{Y})$ into a normed *-algebra as follows: For $f, g \in C_0(\mathcal{Y})$, define

$$f * g(K, x) = \int_K f(K, y) g(K, y^{-1}x) d\mu_K(y),$$

$$f^*(K, x) = \overline{f}(K, x^{-1}) \Delta_K(x^{-1}),$$

$$\|f\| = \sup_{K \in \mathcal{K}(G)} \int_K |f(K, x)| d\mu_K(x).$$

The completion $A_s(G)$ of $C_0(\mathcal{Y})$ with respect to the above norm is a Banach *-algebra called the *subgroup algebra* of G.

For each $K \in \mathcal{K}(G)$, the mapping $f \mapsto f_K$ by $f_K(x) = f(K, x)$ extends to a continuous *-homomorphism, called Φ_K, of $A_s(G)$ into a dense subalgebra of $L^1(K, \mu_K)$.

The following is easy by previous work.

Lemma 116. *For each f in $A_s(G)$, $K \mapsto \|\Phi_K(f)\|$ is continuous on $\mathcal{K}(G)$ and $\|f\| = sup_K \|\Phi_K(f)\|$.*

Definition 117 We define, for $K \in \mathcal{K}(G)$, the space of continuous functions on K as $C(K)$. We are working toward defining a topology on $\cup_{K \in \mathcal{K}(G)} C(K)$, so that a function defined on K is allowed to approach a function defined on K_0 as K approaches K_0 in the topology of closed subgroups as outlined in section 1.4. This will be done in such a fashion that for functions with fixed domain, the topology is that of uniform convergence on compact sets. For a much more detailed account, see [13].

Definition 118 Each unitary representation T of a closed subgroup K can be lifted to a *-representation we denote as $W^{K,T}$ of $A_s(G)$ by $W^{K,T} = T \circ \Phi_K$. From this and Lemma 116, $A_s(G)$ is a reduced Banach *-algebra. Its C^* completion, denoted $C^*_s(G)$, is called the *subgroup C^* algebra* of G, with norm $\|\cdot\|_c$. A representation of the form $W^{K,T}$ is said to be *lifted* from K.

Definition 119

$\mathcal{Q}(G)$ is defined to be the set of all subgroup representation pairs of G, where $\langle H, T\rangle$ is such a pair if $H \in \mathcal{K}(G)$ and $T \in \text{Rep}(H)$. We identify all pairs $\langle K, T\rangle$ where T is identically zero, the resulting element θ being called the *zero element* of $\mathcal{Q}(G)$.

Definition 120

By the *inner hull-kernel* topology of $\mathcal{Q}(G)$, we mean the topology which makes the 1-1 mapping $\langle K, T\rangle \mapsto W^{K,T}$ a homeomorphism with respect to the inner hull kernel topology of $\text{Rep}(A_s(G))$. We follow with some of its notable properties.

Lemma 121. *For $H \in \mathcal{K}(G)$, the topology of $\mathcal{Q}(G)$ relativized to the set of a subgroup-representation pairs whose first coordinate is constant H is just the inner-hull-kernel topology on Rep(H).*

Proof.

This is immediate from the definitions and Lemma 3.3 of [13]. □

Definition 122

Let X be an arbitrary locally compact Hausdorff space, and set $S = X \times \mathbb{C}$. A subset of S is *semicompact* if it is closed and its projection onto X has compact closure, $\mathcal{D}$ we define to be the family of all semicompact subsets of S. As before, $\mathcal{K}(S)$ is the family of closed subsets of S. Let $\mathcal{F}$ be a finite family of nonempty open sets of S and $D \in \mathcal{D}$. Define

$$U(D, \mathcal{F}) = \{A \in \mathcal{K}(S) \mid A \cap D = \emptyset,\ A \cap B \neq \emptyset \text{ for each } B \in \mathcal{F}\}.$$

Definition 123

The *semicompact-open topology* of (S) will be that topology in which the $U(D, \mathcal{F})$, where $D \in \mathcal{D}$ and $\mathcal{F}$ is a finite family of nonempty subsets of S, form a basis of open sets.

Definition 124

Now define $\mathcal{E}(X)$ to be the union of all the spaces $C(A)$, where $A \in \mathcal{K}(S)$, thus the elements of $\mathcal{E}(X)$ are complex-valued functions f such that for the domain of f, $D(f)$, we have $D(f) \in \mathcal{K}(S)$. Identifying a function with its graph we have $\mathcal{E}(X) \subseteq \mathcal{K}(S)$. We will always consider the set $\mathcal{E}(X)$ as being equipped with the (relativized) semicompact-open topology.

Lemma 125. *Suppose $f \in \mathcal{E}(X)$, $\{f_i\}$ is a net of elements of $\mathcal{E}(X)$, and $D(f_i) \to D(f)$ in $\mathcal{K}(X)$. Then $f_i \to f$ in $\mathcal{E}(X)$ if and only if, for each subnet $\{f_i'\}$ of $\{f_i\}$ and for each choice of x_j in $D(f_j')$ (for each j) such that $x_j \to$ some x in $D(f)$, we have $f_j'(x_j) \to f(x)$.*

Proof.

See [13], Lemma 3.2 □

Definition 126

We denote by $\mathcal{E}_s(G)$ the topological space of $\mathcal{E}(G)$ consisting of those f for which $D(f) \subseteq \mathcal{K}(G)$.

Lemma 127. *Let $\{\langle K_i, T_i\rangle\}$ be a net of elements of $\mathcal{Q}(G)$ and $\langle K, T\rangle$ an element of $\mathcal{Q}(G)$. Then $\langle K_i, T_i\rangle \to \langle K, T\rangle$ if and only if, for each finite* *sequence $\phi_1, \cdots, \phi_n$ of functions of positive type on K associated with $\langle K, T\rangle$, and each subnet of $\{\langle K_i, T_i\rangle\}$, there exists (i) a subnet $\{\langle K'^j, T'^j\rangle\}$ of that subnet, and (ii) for each j and each $r = 1, \cdots, n$ a finite sum ϕ_r^j of functions of positive type associated with $\langle K'^j, T'^j\rangle$ such that $\phi_r^i \to \phi_r$ in $\mathcal{E}_s(G)$ for each r.*

Proof.

See [13] Theorem 3.1', page 439. □

We need another result.

Definition 128 Let G a connected, simply connected nilpotent Lie group. By Kirillov, [21], the irreducible representations of G are in one-to-one correspondence with Ad* orbits in $\mathfrak{g}^*$. If $W \in \widehat{G}$, we denote its correspondence orbit by Ω_W.

Lemma 129 (Joy's Lemma). *Let G be a real, connected, simply connected nilpotent Lie group. Let $\langle H_n, S_n\rangle \to \langle H, S\rangle$ in $\mathcal{Q}(G)$. If $f \in \mathfrak{g}^*$ such that $f|_{\mathfrak{h}} \in \Omega_S$, then for every subsequence of $\{\langle H_n, S_n\rangle\}_{n=1}^{\infty}$, there is a subsequence $\{\langle H_{n_i}, S_{n_i}\rangle\}_{i=1}^{\infty}$ such that for each i, there exists $f_i \in \mathfrak{g}^*$ such that $f_i|_{\mathfrak{h}_{n_i}} \in \Omega_{S_{n_i}}$ and $f_i \to f$ in $\mathfrak{g}^*$.*

Proof.

See [19], page 138. □

Induced representations and ideals of $C^*(G,\Omega)$

Discussion 130

In this section we give further specifics about induced representations of transformation group C^* algebras, giving two ways to view them in our case.

Definition 131

In [18], Phil Green defines a covariance algebra to be *quasi-regular* if every primitive ideal lives on a quasi-orbit, for references, see [18], page 221. In the setting of transformation group C^* algebras, quasi-regularity means that for every $P \in \mathrm{Prim}(C^*(G,\Omega))$, $\mathrm{hull}(\mathrm{Res}(P)) = \overline{G \cdot x}$ for some $x \in \Omega$, where we view a closed ideal in $C_0(\Omega)$ as corresponding to a closed set of points. We here comment that all of our algebras are quasi-regular, see [17], we also comment on this fact elsewhere in this thesis.

Discussion 132

We comment that on the topic of induced representations of transformation group C^* algebras that we only here give what we need for the examples in this this paper. We give two different presentations and will use both. More detailed accounts can be found in [18], [25] and [26].

Let G be a group, acting with jointly continuous action on a locally compact Hausdorff space Ω. We will always induce from point evaluations and stabilizers or polarizing subalgebras, which we presuppose to be connected and simply connected.

For $S \subseteq G_x$, the stability group of a point $x \in \Omega$, let τ be a representation of S with Hilbert space H_τ, ρ_x a representation of $C_0(\Omega)$ on H_τ, where for a function $\phi \in C_0(\Omega)$ and a function $f \in H_\tau$ we have $\rho_x(\phi)(f)(r) = \phi(x)f(r)$. We have that (τ, ρ_x) is a covariant pair for $C^*(S,\Omega)$.

We define the *induced representation* of (τ, ρ_x) on $C^*(G, \Omega)$ to be $L = (V, M)$, where $V = \mathrm{ind}_S^G(\tau)$, and M is a representation of $C_0(\Omega)$ on H_V which acts by $M(\phi)(f)(r) = \psi(r \cdot x)f(r)$, we will say more about this at Comment 142. We use the notation

$$L = (V, M) = \mathrm{ind}_{(S,\Omega)}^{(G,\Omega)}(\tau, \rho_x);$$

we will use this notation without reference.

Another way to view induced representations of $C^*(G, \Omega)$ follows. We comment that for our purposes in this paper we will always induce from subgroups contained in stabilizers and point evaluations of Ω, also, our modular functions are always identical 1. However, we here include the modular functions as later on we will prove a result (Proposition 183) which does not depend upon our group G being a nilpotent Lie group. We should here point out that in [18], Phil Green defines $C^*(G, \Omega)$ in terms of a symmetric Haar measure, $\Delta_G^{-1/2}(s)\mathrm{d}\mu_G(s)$. To avoid confusion, we point out that his definitions are mapped to ours by $f \mapsto \Delta_G^{-1/2} f$. This was abstracted from [26], page 340, a more general account may also be found in [18]. We will use this to characterize certain primitive ideals coming from representations living over a fixed point $x \in \Omega$.

We assume that H is a closed subgroup of G.

We assume that we are in the C^* algebra $C^*(G, \Omega)$, and that π is a representation of $C^*(G_x, \Omega)$ acting on the Hilbert space V_π. Let ξ and η be arbitrary vectors in V_π. We define the induced representation, $L = \mathrm{ind}_{(H,\Omega)}^{(G,\Omega)}(\pi)$ as follows:

We first define a $B = C_c(H, \Omega)$-valued inner product on the imprimitivity algebra $C_c(G,\Omega)$ by

$$\langle f,g\rangle_{_B}(t,y)=\gamma_{_H}(t)\int\limits_{s\in G}\overline{f}(s,s\cdot y)g(st,s\cdot y)\mathrm{d}\mu_{_G}(s).$$

The induced representation L acts on the completion of $C_c(G,\Omega)\otimes V_\pi$, completed in the inner product defined by

$$\langle f\otimes\xi,g\otimes\eta\rangle_{_L}=\big\langle\pi(\langle g,f\rangle_{_B})\xi,\eta\big\rangle_{\tau}.$$

The action of $h\in C^*(G,\Omega)$ on the class of $f\otimes\xi$ is given by $(h*f)\otimes\xi$, see [18], page 204, or [26], page 340.

We also will need several more general results.

Proposition 133. *("Induction in Stages") Let $H\supseteq K$ be closed subgroups of G, and L be a *-representation of (K,Ω). Then $ind_{(H,\Omega)}^{(G,\Omega)}(ind_{(K,\Omega)}^{(H,\Omega)}(L))$ is unitarily equivalent to $ind_{(K,\Omega)}^{(G,\Omega)}(L)$.*

Proof.

See [18], Proposition 8 on page 207. □

Proposition 134. *Let H be a subgroup of G. There is a continuous map, Ind_H^G, from $\mathcal{I}(C^*(H,\Omega))$ (ideal space) to $\mathcal{I}(C^*(G,\Omega))$ such that, if L is a representation of $C^*(H,\Omega)$, then $Ind_H^G(ker(L))=ker(Ind_{(H,\Omega)}^{(G,\Omega)}(L))$.*

Proof.

This originally appears in [18], Proposition 9 on pages 208-209, see also [26], Lemma 3.5 on page 343. □

Proposition 135. *The process of induction on transformation group C^* algebras preserves weak containment, that is to say, if H is a subgroup of*

G, and L_1 and L_2 are representations of $C^*(H,\Omega)$ and $L_1 \prec L_2$, then $ind_{(H,\Omega)}^{(G,\Omega)}(L_1) \prec ind_{(H,\Omega)}^{(G,\Omega)}(L_2)$.

Proof.

Note that weak containment may be viewed as a statement about containment of kernels, see Definition 6, and appeal to the Proposition 134 above. □

Proposition 136. *Let G be a group acting on a locally compact Hausdorff space Ω. Let x be a fixed point in Ω, and τ a representation of G_x. If τ is irreducible, then the induced representation of $C^*(G,\Omega)$,*

$$L = ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau, \rho_x),$$

is an irreducible representation of $C^*(G,\Omega)$.

Proof.

See [26], Proposition 4.2, page 344. □

Definition 137

Let (V,M) be a representation of (H,Ω), $s \in G$. Let $K = sHs^{-1}$, and let $(V,M)^s$ be the covariant representation of (K,Ω) given by

$$V^s(r) = V(s^{-1}rs) \qquad M^s(\phi) = M({}^s\phi).$$

Recall that ${}^s\phi(x) = \phi(s^{-1}\cdot x)$, see Definition 1.

Comment 138

The system (G,Ω) is presupposed to satisfy the second axiom of countability for the next lemma, which is true by hypothesis in our case.

Lemma 139. *If* $S = ind_{(H,\Omega)}^{(G,\Omega)}(V,M)$ *and* $T = ind_{(K,\Omega)}^{(G,\Omega)}((W,N))$*, then* $S \cong T$ *if and only if for some* $s \in G$*, we have* $T = ind_{(sKs^{-1},\Omega)}^{(G,\Omega)}((V,M)^s)$.

Proof.

See [15], Theorem 2.1. □

Section 1.11

A topology on representations of subgroup C^* algebras

Discussion 140

In this section we do a construction which is Siegfried Echterhoff's construction in [7] modified for our particular situation, and use these tools to prove an important lemma that we will need later on.

Definition 140

Let G be our connected, simply-connected nilpotent Lie group, and again denote by $\mathcal{K}(G)$ the space of closed subgroups of G endowed with the compact Hausdorff topology of Fell in [10]. Let $\mathcal{N}$ be a locally compact space. Assume that $H : \mathcal{N} \mapsto \mathcal{K}(G);\ H \mapsto H_i$ is a continuous map from $\mathcal{N}$ to $\mathcal{K}(G)$. Define:

1) $\mathcal{N}^H = \{(i,x) \in \mathcal{N} \times G \mid x \in H_i\}$

2) $\mathcal{N}^{2H} = \{(i,x,y) \in \mathcal{N} \times G \times G \mid x,y \in H_i\}$

3) On $\mathcal{N} \times G \times G$ we introduce and equivalence relation " $\sim$ " by

$$(i,x,y) \sim (i',x',y') \iff i=i',\ x=x' \text{ and } y \in y'H_i.$$

The H - *equivalence class* of some $(i,x,y) \in \mathcal{N}\times G\times G$ we denote by $(i,x,\dot{y})$.

4) If $K : \mathcal{N} \mapsto \mathcal{K}(G)$ is another continuous map such that $H_i \subseteq K_i$ for all $n \in \mathcal{N}$, then $\mathcal{N}^{2K}$ is a union of H- equivalence classes and we define $\mathcal{N}_H^K$ to be the quotient space $\mathcal{N}^{2K}/\sim$

Comment 142

We here comment that in all of our examples in this thesis we will be using $\mathcal{N} = \mathbb{N}\cup\infty$, the one-point compactification of the natural numbers.

Lemma 143. *$\mathcal{N}^H$, $\mathcal{N}^{2H}$ and $\mathcal{N}_H^K$ are locally compact Hausdorff spaces and the quotient map $q:\ \mathcal{N}^{2K} \mapsto \mathcal{N}_H^K$ is open.*

Proof.

See [7], Lemma 1 on page 64. $\square$

Definition 144

Now let G again be a nilpotent Lie group and (G,Ω) denote a covariant system. We will make the spaces $C_c(\mathcal{N}^H, C_0(\Omega))$ and $C_c(\mathcal{N}_H^K, C_0(\Omega))$ into normed *- algebras. We comment that we simplify Echterhoff's work slightly as modular functions are identical 1 for a nilpotent Lie group. Define multiplication, involution, and norms on these spaces by

$$f*g(i,t,x) = \int\limits_{s\in H_i} f(i,s,x)g(i,s^{-1}t,s^{-1}\cdot x)\mathrm{d}\mu_{H_i}(s)$$

$$f^*(i,t,x) = \overline{f}(i,t^{-1},t^{-1}\cdot x)$$

$$\|f\|_1 = \sup_{i\in\mathcal{N}} \int\limits_{s\in H_i} \sup_{x\in\Omega} |f(i,s,x)|\mathrm{d}\mu_{H_i}(s)$$

for $f,g \in C_c(\mathcal{N}^H, C_0(\Omega))$, and also set

$$F * G(i,t,\dot{u},x) = \int\limits_{s\in H_i} F(i,s,\dot{u},x)G(i,s^{-1}t,s^{-1}\dot{u},s^{-1}\cdot x)\mathrm{d}\mu_{H_i}(s)$$
$$F^*(i,t,\dot{u},x) = \overline{F}(i,t^{-1},t^{-1}\dot{y},t^{-1}\cdot x)$$
$$\|F\|_1 = \sup_{i\in\mathcal{N}} \int\limits_{s\in K_i} \sup_{y\in K_i}\left\{\sup_{x\in\Omega}|F(i,s,\dot{y},x)|\right\}\mathrm{d}\mu_{H_i}(s)$$

for $F, G \in C_c(\mathcal{N}_H^K, C_0(\Omega))$. We have defined multiplication and involution "fiberwise" on the "fibers" $C_c(i, C_0(\Omega))$ and $C_c(i, C_c(K_i/H_i, C_0(\Omega)))$.

Theorem 145. *The spaces $C_c(\mathcal{N}^H, C_0(\Omega))$ and $C_c(\mathcal{N}_H^K, C_0(\Omega))$ are normed * - algebras if we define multiplication, involutions, and norms as above.*

Proof.

See [7], Theorem 1 on page 65.

Definition 146

We denote by $L^1(\mathcal{N}^H, C_0(\Omega))$ and $L^1(N_H^K, C_0(\Omega))$ to be the completion of $C_c(\mathcal{N}^H, C_0(\Omega))$ and $C_c(\mathcal{N}_H^K, C_0(\Omega))$ with respect to the above norms.

Again, for any $F \in C_c(\mathcal{N}_H^K, C_0(\Omega))$ we denote by F_i the element of $C_c(K_i, C_0(K_i/H_i, C_0(\Omega)))$ defined by $F_i(t,\dot{u},x) = F(i,t,\dot{u},x)$. For each $n \in \mathcal{N}$ the map $F \mapsto F_i$ extends to a norm-decreasing * - homomorphism from $L^1(\mathcal{N}_H^K, C_0(\Omega))$ into a dense subalgebra of $L^1(K_i, C_0(K_i/H_i, C_0(\Omega)))$. Hence any covariant representation τ of the system $\big(K_i, C_0(K_i/H_i, C_0(\Omega))\big)$ defines a * - representation of $L^1(\mathcal{N}_H^K, C_0(\Omega))$ by

$$(i,\tau)(F) = \tau(F_i)$$

We may similarly look at pairs (i, ρ), $i \in \mathcal{N}$, and $\rho \in \text{Rep}(H_i, C_0(\Omega))$ as * - representations of $L^1(\mathcal{N}^H, C_0(\Omega))$. It is clear that these representations separate the elements of $L^1(\mathcal{N}_H^K, C_0(\Omega))$ and $L^1(\mathcal{N}^H, C_0(\Omega))$, respectively. We now define our subgroup C^* algebras.

Definition 147

Let $C^*(\mathcal{N}^H, C_0(\Omega))$ and $C^*(N_H^K, C_0(\Omega))$ be the enveloping C^* algebras of $L^1(\mathcal{N}^H, C_0(\Omega))$ and $L^1(\mathcal{N}_H^K, C_0(\Omega))$, respectively. We call $C^*(\mathcal{N}^H, C_0(\Omega))$ the *subgroup algebra* of $(G, C_0(\Omega))$ and $C^*(\mathcal{N}_H^K, C_0(\Omega))$ the *imprimitivity subgroup algebra* of $(G, C_0(\Omega))$. Furthermore, the representation spaces

$$\mathcal{R}(C^*(\mathcal{N}_H^K, C_0(\Omega))) = \left\{(i, \pi) \mid i \in \mathcal{N},\ \pi \in \text{Rep}(K_i, C_0(K_i/H_i, C_0(\Omega)))\right\}$$

and

$$\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))) = \left\{(i, \rho) \mid i \in \mathcal{N}, \rho \in \text{Rep}(H_i, C_0(\Omega))\right\}$$

equipped with the relative topologies of $\text{Rep}(C^*(\mathcal{N}_H^K, C_0(\Omega)))$ and $\text{Rep}(C^*(\mathcal{N}^H, C_0(\Omega)))$ are the *subgroup representation spaces* of $C^*(\mathcal{N}_H^K, C_0(\Omega))$ and $C^*(\mathcal{N}^H, C_0(\Omega))$, respectively.

Proposition 158. *Every irreducible representation of $C^*(\mathcal{N}_H^K, C_0(\Omega))$ is a subgroup representation. The same is true of $C^*(\mathcal{N}^H, C_0(\Omega))$.*

Proof.

See [7], Proposition 2 on page 66.

Proposition 149. *Let* $(G, C_0(\Omega))$ *be a covariant system,* $\mathcal{N}$ *be a locally compact space and* $H,\ K : \mathcal{N} \mapsto \mathcal{K}(G);\ i \mapsto H_i,\ K_i$ *be continuous maps such that* $H_i \subseteq K_i$ *for all* $i \in \mathcal{N}$*. Then the map*

$$Ind_H^K :\ \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))) \mapsto \mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)));\ (i, \rho) \mapsto (i, ind_{H_i}^{K_i}(\rho))$$

is continuous.

Proof.

See [7], Proposition 6, page 69. □

Now we look at restrictions of representations to subgroups.

Proposition 150. *Let* $(G, C_0(\Omega)),\ \mathcal{N},\ H,$ *and* K *be as in the last proposition. Then the map*

$$Res_H^K :\ \mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega))) \mapsto \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)));\ (i, \pi) \mapsto (i, \pi|_{(H_i, \Omega)})$$

is continuous.

Proof.

See [7], Proposition 7 on page 70. □

Comment 151

As the last proposition will be important, we look at it a bit closer in the situation where we will need it. Define $\mathcal{N} = \mathbb{N} \cup \infty$ and assume that we have two continuous maps H and K from $\mathcal{N}$ to $\mathcal{K}(G)$, with $H(n) = H_n$ and $H(\infty) = H_\infty$, similar for the map K, with H_i codimension one in K_i for all $i \in \mathcal{N}$. Assume that we have a collection $\big\{\{\pi_i\} \mid i \in \mathcal{N}\big\}$, each π_n a representation of $C^*(K_n, \Omega)$, and π_∞ is a representation of $C^*(K_\infty, \Omega)$.

Assume that in the representation space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$ we have that $(n, \pi_n) \to (\infty, \pi_\infty)$. The previous proposition says that the restriction map $\mathrm{Res}_H^K : \mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega))) \mapsto \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ by $\mathrm{Res}_H^K(n, \pi_n) = (n, \pi_n|_{(H_n,\Omega)})$ is continuous, i.e. in the space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ we have $(n, \pi_n|_{(H_n,\Omega)}) \to (n, \pi_\infty|_{(H_\infty,\Omega)})$.

Comment 152

We need the following result of Dana Williams in the proof of Proposition 154 below. We comment that the original has a small (non-fatal) typo which we fix.

Proposition 153. *Let H be a normal subgroup of G and L a representation of $C^*(H, \Omega)$. Then*

$$ker(Res_H^G(Ind_H^G(L))) = \bigcap_{s \in G} ker(L^s)$$

Proof.

See [26], Proposition 5.5 on page 354. □

Proposition 154. *Assume that G is a nilpotent Lie group, Ω is a locally compact Hausdorff space upon which G acts via bicontinuous automorphisms, and we have two continuous maps K and H from $\mathcal{N} = \mathbb{N} \cup \infty$ to $\mathcal{K}(G)$ with H_n codimension 1 in K_n for all n. We also assume that we have a collection $\{\{(i, \pi_i)\} \mid i \in \mathcal{N}\}$, each of the π_i's induced from (H_i, Ω), and each π_i is irreducible on $C^*(K_i, \Omega)$. We further assume that in the space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$ we have $(n, \pi_n) \to (\infty, \pi_\infty)$. Then we may choose a collection*

$$\{\{(n_i, \rho_{n_i})\} \mid i \in \mathcal{N}\}$$

with each ρ_{n_i} *being an irreducible representation of* $C^*(H_{n_i}, \Omega)$ *and* ρ_∞ *an irreducible representation of* (H_∞, Ω)*, these representations satisfying:*

$$\pi_{n_i} \cong ind_{(H_{n_i},\Omega)}^{(K_{n_i},\Omega)}(\rho_{n_i}),$$

$$\pi_\infty \prec Ind_{(H_\infty,\Omega)}^{(K_\infty,\Omega)}(\rho_\infty),$$

and

$$(n_i, \rho_{n_i}) \to (\infty, \rho_\infty)$$

in $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$*.*

Proof.

We have assumed that π_n is induced from (H_n, Ω), therefore we may assume that $\pi_n = \mathrm{ind}_{(H_n,\Omega)}^{(K_n,\Omega)}(\rho_n)$. By the continuity of restriction (Proposition 150, see also Comment 151), we have:

$$(n, \pi|_{(H_n,\Omega)}) \to (\infty, \pi|_{(H_\infty,\Omega)}),$$

in $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$. For each n the restricted representation is equivalent to the following direct integral:

$$(n, \pi_n|_{(H_i,\Omega)}) \cong (n, \int_{s\in K_n/H_n}^{\oplus} \rho_n^s \ ) \tag{1}$$

remembering that H_n is normal in K_n, see again Proposition 153 and Definition 137 for notation. As $\int_{s\in K_n/H_n}^{\oplus} \rho_n^s$ represents an ideal of $C^*(H_n, \Omega)$ and the the sequence of ideals represented by the representations in formula (1) above converges to the ideal represented by $(\infty, \pi_\infty|_{(H_\infty,\Omega)})$, we may use

Proposition 22 to choose an irreducible representation ρ'_∞ of (H_∞, Ω) with $\rho'_\infty \prec \pi_\infty|_{(H_\infty,\Omega)}$ and $\pi_\infty \prec \operatorname{ind}_{(H_n,\Omega)}^{(K_n,\Omega)}(\rho'_\infty)$ and a subsequence $\{(n_i, \rho'_{n_i}\}_{i=1}^{\infty})$ with $\rho'_{n_i} = \rho^{s_{n_i}}$ for some $s_{n_i} \in K_{n_i}$ for all n_i, and $(i, \rho'_{n_i}) \to (\infty, \rho'_\infty)$ in the representation space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$. Notably, we have

$$\operatorname{ind}_{(H_{n_i},\Omega)}^{(K_{n_i},\Omega)}(\rho'_{n_i}) \cong \pi_{n_i},$$

as by Proposition 139 we have

$$\pi_{n_i} \cong \operatorname{ind}_{(H_{n_i},\Omega)}^{(G,\Omega)}(\rho_{n_i}) \cong \operatorname{ind}_{(H_{n_i},\Omega)}^{(G,\Omega)}(\rho_{n_i}^{s_{n_i}}).$$

We also have

$$\pi_\infty \prec \operatorname{ind}_{(H_\infty,\Omega)}^{(K_\infty,\Omega)}(\rho'_\infty),$$

as we have previously noted. □

Lemma 155. *Assume that G is a nilpotent Lie group, Ω is a locally compact Hausdorff space upon which G acts via bicontinuous automorphisms, and we have a continuous map H from $\mathcal{N} = \mathbb{N} \cup \infty$ to $\mathcal{K}(G)$ with $H(n) = H_n$ and $H_n \to H_\infty$ in $\mathcal{K}(G)$. We also assume that we have a collection $\{\{\pi_i\} \mid i \in \mathcal{N}\}$, with $\pi_n = (\chi_{f_n}, M_{x_n})$ and $\pi_\infty = (\chi_{f_\infty}, M_{x_\infty})$, one-dimensional representations of $C^*(H_n, \Omega)$ and $C^*(H_\infty, \Omega)$, the χ's characters and the M's point evaluations of Ω.*

We assume that in the space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ we have $(n, \pi_n) \to (\infty, \pi_\infty)$. Then by passing to a subsequence, we may choose another collection $\{\{f'_n\}_{n=1}^{\infty}, f'\}$ such that

$$f'_n \to f_\infty \quad \textit{and} \quad x_n \to x$$

and

$$\pi_n = (\chi_{f'_n}, M_{x_n})$$

Proof.

Now we note that for any $\phi \in C_c(G)$ and $\psi \in C_c(\Omega)$ we may consider $\phi \cdot \psi \in C^*(\mathcal{N}^H, C_0(\Omega))$ by setting $(\phi \cdot \psi)(i) = \phi|_{H_i} \cdot \psi$. We note that

$$\pi_n(\phi|_{H_n} \cdot \psi) = \psi(x_n) \int_{s \in H_n} \phi(s)\chi_{f'_n}(s)\mathrm{d}\mu_{H_n}(s)$$
$$\longrightarrow \psi(x) \int_{s \in H} \phi(s)\chi_{f'}(s)\mathrm{d}\mu_H(s) = \pi_\infty(\phi|_{H_\infty} \cdot \psi).$$

Let e denote the identity element of the group. Since the restriction map from $\mathcal{R}(C^*(\mathcal{N}, C_0(\Omega)))$ to Ω, Res_e^H, is continuous (Proposition 150), we have $M_{x_n} \to M_x$ and $x_n \to x$.

We choose ψ to be identical one in a neighborhood of $x \in \Omega$, so that it "goes away" in the last displayed equation. By the work of Fell in [12] and explicated in section 1.9 of this thesis, and the equivalence of the representation spaces of G and $\widehat{G}$ described by Fell in [11], we get that in the subgroup-pair topology (again, section 1.9 of this thesis) that

$$\langle \chi_{f'_n}, H_n \rangle \to \langle \chi_{f'}, H_\infty \rangle,$$

and we may pass to a subsequence and use Joy's Lemma (Lemma 129) to find a sequence $\{f'_n\}_{n=1}^\infty \subseteq \mathfrak{g}^*$ with $\chi_{(f'_n|_{H_n})} = \chi_{(f_n|_{H_n})}$ and $f'_n \to f$ in $\mathfrak{g}^*$. $\square$

Comment 156

The following lemma is very important in Chapter 2, and what we have been building up to in this entire section.

Lemma 157. *Assume that we have a collection* $\{P, \{P_n\}_{n=1}^{\infty}\}$ *of primitive ideals of* $C^*(G,\Omega)$*, with*

$$P_n = ker(L_n) = ker(ind_{(G_{x_n},\Omega)}^{(G,\Omega)}(\tau_{f_n,x_n}, M_{x_n})) \longrightarrow P = ker(L) = ker(ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, M_x)).$$

We may choose subsequences

$$\{y_{n_i}\}_{i=1}^{\infty} \subseteq \Omega \text{ and } \{f'_{n_i}\}_{i=1}^{\infty} \subseteq \mathfrak{g}^* \quad \text{with } y_{n_i} \to x' \in \Omega, f'_{n_i} \to f',$$

with

$$ker(L_{n_i}) = ker(ind_{(G_{y_{n_i}},\Omega)}^{(G,\Omega)}(\tau_{f'_{n_i},y_{n_i}}, M_{y_{n_i}}))$$

and

$$ker(L) = ker(ind_{(G_{x'},\Omega)}^{(G,\Omega)}(\tau_{f'}, M_{x'})).$$

Proof.

If an infinite subcollection of $\{L_n\}_{n=1}^{\infty}$ are 1-dimensional representations of $C^*(G,\Omega)$, we may use the constant sequence $H_n = G$ for all n in Lemma 155 just proven.

Thus (by passing to a subsequence if necessary) we assume that all of the L_n's are not one-dimensional, hence are induced from codimension one subgroups. So assume that for all n we have

$$L_n \cong \mathrm{ind}_{(H_n,\Omega)}^{(G,\Omega)}(\rho_n),$$

with H_n codimension 1 in G. We may assume by passing to a subsequence if necessary that $H_n \to H_\infty$ in $\mathcal{K}(G)$, as $\mathcal{K}(G)$ is compact. So the map H defined on $\mathcal{N} = \mathbb{N} \cup \infty$ by $H(n) = H_n$ and $H(\infty) = H_\infty$ is a continuous map from $\mathcal{N} = \mathbb{N} \cup \infty$ to $\mathcal{K}(G)$.

We do the codimension one case of

$$L_n \cong \operatorname{ind}_{(H_n,\Omega)}^{(G,\Omega)}(\chi_{f_n}, M_{x_n}),$$

with each χ_{f_n} a character of the codimension one subgroup H_n, and $H_n \to H_\infty$ in $\mathcal{K}(G)$. We may apply pass to a subsequence and apply Proposition 154 to produce a collection $\{\rho_i \mid i \in \mathcal{N}\}$ of the collection $\{(H_i, \Omega) \mid i \in \mathcal{N}\}$, such that for each index $i \in \mathcal{N}$ we have $\rho_i = (\chi_{f_i'}, M_{x_i'})$.

We have for each n,

$$L_n \cong \operatorname{ind}_{(H_n,\Omega)}^{(G,\Omega)}(\chi_{f_n'}, M_{x_n'})$$

and

$$L_\infty \prec \operatorname{ind}_{(H_\infty,\Omega)}^{(G,\Omega)}(\chi_{f_\infty'}, M_{x_\infty'}).$$

As we may assume that $(n, \rho_n) \to (\infty, \rho_\infty)$ in $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ we may apply Lemma 155 and pass to a subsequence to conclude that f_n' converges in $\mathfrak{g}^*$ to f and x_n' converges to x_∞' in Ω.

In general, we assume that we have the following

$$L_n \cong \operatorname{ind}_{(K_n,\Omega)}^{(G,\Omega)}(L_n')$$

and

$$L_\infty \prec \operatorname{ind}_{(K_\infty,\Omega)}^{(G,\Omega)}(L'_\infty),$$

and by passing to a subsequence we may assume that all of the K_i's have the same codimension in G and $K_n \to K_\infty$.

We may assume by multiple applications of Proposition 154 that $(n, L'_n) \to (\infty, L'_\infty)$ in $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$. Our inductive hypothesis is that when each K_i is codimension l and each L'_i is a one-dimensional representation of $C^*(K_i, \Omega)$ that we may assume $(n, L'_n) \to (\infty, L'_\infty)$ in the representation space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$ and can produce the desired sequence of the conclusion of the lemma.

We note that when $l = 0$ or 1 that we have established this and the final conclusion of the lemma.

When no infinite subcollection of $\{L'_n\}_{n=1}^\infty$ is one-dimensional, we will define a sequence $\rho_n = (\tau_{f'_n}, M_{x'_n})$ on the codimension of K_n in G such that $(n, \rho_n) \to (\infty, \rho_\infty)$ in $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ inductively.

Now, we assume that we have a collection $\big\{\{L'_i\} \mid i \in \mathcal{N}\big\}$ with

$$L_n \cong \operatorname{ind}_{(K_n,\Omega)}^{(G,\Omega)}(L'_n)$$

and

$$L \prec \operatorname{ind}_{(K_\infty,\Omega)}^{(G,\Omega)}(L'_\infty).$$

We also assume

$$L'_n \cong \operatorname{ind}_{(H_n,\Omega)}^{(K_n,\Omega)}(\tau_{f_n}, M_{x_n}),$$

$$L'_\infty \cong \operatorname{ind}_{(H_\infty,\Omega)}^{(K_\infty,\Omega)}(\tau_{f_\infty}, M_{x_\infty})$$

We also assume that each H_n codimension $l+1$ in G, and each K_n codimension l in G, and we may assume by passing to a subsequence if necessary that $H_n \to H_\infty$ and $K_n \to K_\infty$. We assume again by multiple applications of Lemma 154 that in the representation space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$ that we have $(n, L'_n) \to (\infty, L'_\infty)$.

We may apply Proposition 154 to produce a collection $\{\{\rho_i\} \mid i \in \mathcal{N}\}$ with $\rho_n = (\tau_{f'_n}, M_{x'_n})$ and $\rho_\infty = (\tau_{f'_\infty}, M_{x_\infty})$ of the collection $\{(H_i, \Omega) \mid i \in \mathcal{N}\}$. We note that we may also assume

$$L_n \cong \operatorname{ind}_{(H_n,\Omega)}^{(G,\Omega)}(\rho_n)$$

and

$$L_\infty \prec \operatorname{ind}_{(H_\infty,\Omega)}^{(G,\Omega)}(\rho_\infty).$$

We note that we may assume that in the representation space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ that we have $(n, \rho_n) \to (\infty, \rho_\infty)$. We further note that H_n is codimension $l+1$ in G; we may repeat this step until we have a sequence of one-dimensional representations, and by applying Lemma 155, we are done by induction. $\square$

Comment 158

We here comment that we have slighty abstracted the work of Joy in [19], and the reverse implication is trivial.

Chapter 2

The topology on $\mathrm{Prim}(C^*(G,\Omega))$

Discussion 159

In this chapter we describe the topology of the primitive ideal space of our transformation group C^* algebra $C^*(G,\Omega)$ given that G is a connected, simply connected nilpotent Lie group and Ω/G is a T_0 space. We will define a map from a quotient space of $\mathfrak{g}^* \times \Omega$ to $\mathrm{Prim}(C^*(G,\Omega))$ that is a homeomorphism, analogous to what Kirillov and Brown did in [21] and [1], see also [19] which is a nice proof of Brown's result. Our result follows some of the methods used by Dana Williams' in [26].

Discussion 160

Again we state the hypotheses: G is a connected, simply connected nilpotent Lie group, and Ω a locally compact Hausdorff space. We also require that all stability subgroups be connected, by Corollary 40 this means that they are closed and simply connected.

Definition 161

We must make several definitions, which will be used repeatedly for the remainder of this paper. For $x \in \Omega$, $f \in \mathfrak{g}^*$, let G_x denote the stabilizer subgroup of x in G, let $\mathfrak{p}_x$ denote a polarizing subalgebra of $\mathfrak{g}_x$ with respect to the action of a given f, let $\mathfrak{p}$ denote any subalgebra of $\mathfrak{g}_x$ which is isotropic for the restriction of f to $\mathfrak{g}_x$, see Definition 83. We remind the reader of Definition 107, that for $\mathfrak{s}$ any Lie subalgebra of a nilpotent Lie algebra $\mathfrak{g}$, by $\mathfrak{s}^\perp$ we mean the set of all linear functionals in $\mathfrak{g}^*$ which vanish on $\mathfrak{s}$. We also define

$\chi_{f,P}$ to be the obvious character (see Corollary 85) of $P = \exp(\mathfrak{p})$,

χ_{f,P_x} to be the obvious character of $P = \exp(\mathfrak{p}_x)$,

$\tau_{f,x} = \mathrm{ind}_{P_x}^{G_x}(\chi_{f,P_x})$, an irreducible representation of G_x,

$\tau'_{f,x} = \mathrm{ind}_{P}^{G_x}(\chi_{f,P})$, a representation of G_x, not in general irreducible.

Definition 162

Let (V, M) be a covariant pair for (G, Ω). If M is induced from a point evaluation ρ_x of a point $x \in \Omega$, i.e., if for $\phi \in C_0(\Omega)$ we have $M(\phi)(r) = \phi(r \cdot x)$, we often write (V, M_x) for (V, M).

Comment 163

We observe that $L = (V, M) = (V, M_x)$ acts on a function f in $C^*(G, \Omega)$ and a function h in the Hilbert space of V as follows:

$$L(f)(h)(r) = M(V(f)h)(r) = M(V\big(f(\,\cdot\,, \cdot x)h\big))(r) =$$

$$\int_{s\in G} f(s, r\cdot x)h(s^{-1}r)\mathrm{d}\mu_G(s).$$

We also comment that on functions of the form $\psi \cdot \phi$, for $\psi \in C_0(\Omega)$, $\phi \in L^1(G)$, the L^1 functions of G, L acts as

$$L(\psi \cdot \phi)h(r) = \psi(r \cdot x) \int\limits_{s\in G} \phi(s)h(s^{-1}r)\mathrm{d}s.$$

The second observation about L will be of importance in Chapter 3.

Lemma 164. *Let $x \in \Omega$, $f \in \mathfrak{g}^*$ be given.*

Let $\mathfrak{p}_x$ be polarizing for the restriction of f to $\mathfrak{g}_x$, and let $\mathfrak{p}$ be isotropic (not necessarily polarizing) for the restriction of f in $\mathfrak{g}_x$. Then $L' = (V', M) = \mathrm{ind}_{(P,\Omega)}^{(G,\Omega)}(\chi_{f,P}, x)$ weakly contains $L = (V, M) = \mathrm{ind}_{(P_x,\Omega)}^{(G,\Omega)}(\chi_{f,P_x}, x)$.

Proof.

By Lemma 108, the representation $\tau'_{f,x} = \mathrm{ind}_P^{G_x}(\chi_{f,P})$ weakly contains $\tau_{f,x} = \mathrm{ind}_{P_x}^{G_x}(\chi_{f,P_x})$. Let $\phi \in C_0(\Omega)$. Let M_x be the representation of $C_0(\Omega)$ by $M_x(\phi) = \phi(x)$ as a multiplication operator. As M_x is a multiplication operator and factors through inner products, we have that as covariant pairs of $C^*(G_x, \Omega)$ that $(\tau'_{f,x}, M_x)$ weakly contains $(\tau_{f,x}, M_x)$. As induction on $C^*(G, \Omega)$ preserves weak containment (See Proposition 135), the conclusion is clear from this and "induction in stages", see Proposition 133. $\square$

Observation 165

We observe that by the discussion in Definition 3, and using the terminology of Lemma 164, that we have

$$\ker(L) \supseteq \ker(L'.)$$

This observation and the next will be important in the proof of Lemma 168 below.

Definition 166

Let $f \in \mathfrak{g}^*$, and $\mathfrak{p}$ be isotropic for the functional f, and χ_f denote the expected character of $P = \exp(\mathfrak{p})$. Let $W = \mathrm{ind}_P^G$ denote the induced representation of the group acting on the space H. Let $h \in C_c(G)$. We define an averaging map A over G/P cosets by

$$A(h)(g) = \int\limits_{s \in P} \chi_{f,P}(s) h(gs) \mathrm{d}\mu_P(s).$$

Note, for $m \in P$ we have

$$A(h)(gm) = \int\limits_{s \in P_n} \chi_{f,P}(s) h(gms) \mathrm{d}\mu_{P_n}(s) =$$

$$(\text{by } s \to m^{-1}s)$$

$$\int_{s\in P_n} \chi_{f,P}(m^{-1}s)h(gs)\mathrm{d}\mu_{P_n}(s) = \chi_{f,P}(m^{-1})A(h)(g),$$

giving us $A(h) \in H$, see Definition 74. Notably, the range of $C_c(G)$ under the action of A is dense in H.

Definition 167

Now define $\phi : \mathfrak{g}^* \times \Omega \mapsto \mathrm{Prim}(C^*(G, \Omega))$ by

$$\phi(f, x) = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, \rho_x)).$$

Lemma 168. *ϕ is continuous in the product topology of $\mathfrak{g}^* \times \Omega$.*

Proof.

Let $F = \{I \in \mathrm{Prim}(C^*(G, \Omega)) \mid I \supseteq J\}$ be an arbitrary closed subset of $\mathrm{Prim}(C^*(G, \Omega))$. Assume also that $\{(f_n, x_n)\}_{n=1}^{\infty} \subseteq \mathfrak{g}^* \times \Omega$ converges to (f, x) in the product topology of $\mathfrak{g}^* \times \Omega$, and $\phi(f_n, x_n) \supseteq J$ for all n. We must show that $\phi(f, x) \supseteq J$.

As the space of closed subgroups of G, $\mathcal{K}(G)$, is compact, assume $G_{x_n} \to S$. By Proposition 65 $S \subseteq G_x$. Also defining $\mathfrak{p}_n$ to be a polarizing subalgebra of G_{x_n} with respect to the restriction of f to $\mathfrak{g}_x$, assume that $\mathfrak{p}_n \to \mathfrak{p} \subseteq \mathfrak{s}$. We comment that $\mathfrak{p}$ is isotropic for f in $\mathfrak{g}_x$, but not in general polarizing. We must make several definitions before we get to the main proof. Define $(\chi_{f_n,P_n}, M_{x_n})$ on the pair (P_n, Ω), acting on the Hilbert space $\mathbb{C}$, by

$$\chi_{f_n,P_n}(s)(z) = e^{i\cdot\log(s)}z, \quad M_{x_n}(\phi)(z) = \phi(x_n)z, \quad \text{for } \phi \in C_0(\Omega),\ z \in \mathbb{C};$$

we note that for and $s \in P_{x_n}$ that $\phi(s \cdot x_n) = \phi(x_n)$ as $P_{x_n} \subseteq G_{x_n}$, and a similar statement is true for P and P_x.

Further define:

$$L_n = \mathrm{ind}_{(P_x,\Omega)}^{(G,\Omega)}(\chi_{f_n,P_n}, x_n), \text{ the } n_{th} \text{ induced representation of } C^*(G,\Omega),$$
$$L_\infty = \mathrm{ind}_{(P,\Omega)}^{(G,\Omega)}(\chi_{f,P}, x),$$
$$L = \mathrm{ind}_{(P_x,\Omega)}^{(G,\Omega)}(\chi_{f,P_x}, x),$$

We need to induce to the C^* algebra $C^*(G,\Omega)$. Define (V_n, M_n) on the n_{th} Hilbert space, H_n, where

$$H_n = \Bigg\{ f : G \mapsto \mathbb{C} \mid f(sp) = \chi_{f_n,P_n}(p^{-1})f(s),\ p \in P_n,\ s \in G/P_n \text{ and}$$
$$\int\limits_{\dot{s}\in G/P_n} \|f(\dot{s})\|^2 \mathrm{d}\mu_{G/P_n}(\dot{s}) < \infty \Bigg\},$$

by

$$V_n(s)(f)(r) = f(s^{-1}r), \quad M_n(\phi)(f)(r) = \phi(r \cdot x_n)f(r),$$
$$\text{where } f \in H_n,\ \phi \in C_0(\Omega),\ s, r \in G.$$

Now we define, for $g \in C_c(G)$, and $r \in G$

$$A(g)(r) = \int\limits_{s\in P} \chi_{f,P}(s)g(rs)\mathrm{d}\mu_P(s)$$

and

$$A_n(g)(r) = \int\limits_{s\in P_n} \chi_{f_n,P_n}(s)g(rs)\mathrm{d}\mu_{P_n}(s),$$

see Definition 166.

We claim that it suffices to show that for all $h \in C_c(G,\Omega)$ and $g \in C_c(G)$ that $\langle L_n(h)A_n(g), A_n(g)\rangle \to \langle L_\infty(h)A(g), A(g)\rangle$. Assume that this is true. As the image of $C_c(G)$ under A (resp. A_n) is dense in H (resp. H_n), we have that if $h' \in J$, as $L_n(h') = 0$ for all n, we have $L_\infty(h') = 0$.

Now we show

$$\langle L_n(h)A_n(g), A_n(g)\rangle \to \langle L_\infty(h)A(g), A(g)\rangle$$

for any $h \in C_c(G,\Omega)$, any $g \in C_c(G)$. Now by Lemma 4, see also Comment 163, we have that

$$L_n(h)(A_n(g))(r) =$$

$$\int_{s\in G} M_n(h(s,\cdot))(V_n(s)(A_n(g))(s))(r)\mathrm{d}\mu_G = \int_{s\in G} h(s, r\cdot x_n)A_n(g)(s^{-1}r)\mathrm{d}\mu_G(s) =$$

$$\int_{s\in G}\int_{t\in P_n} h(s, r\cdot x_n)\chi_{f_n,P_n}(t)g(s^{-1}rt)\mathrm{d}\mu_{P_n}(t)\,\mathrm{d}\mu_G(s),$$

and when we take an inner product we have

$$\langle L_n(h)A_n(g), A_n(g)\rangle =$$

$$\int_{\dot{r}\in G/P_n}\int_{s\in G} h(s,\dot{r}\cdot x_x)(A_n(g))(s^{-1}\dot{r})\overline{(A_n(g))(\dot{r})}\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_{G/P_n}(\dot{r}) =$$

$$\int_{\dot{r}\in G/P_n}\int_{s\in G}\int_{t\in P_n}\int_{u\in P_n} h(s,\dot{r}\cdot x_n)\chi_{f_n,P_n}(t)g(s^{-1}\dot{r}t)\overline{\chi_{f_n,P_n}(u)}\;\overline{g(\dot{r}u)}\cdot$$

$$\mathrm{d}\mu_{P_n}(u)\,\mathrm{d}\mu_{P_n}(t)\,\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_{G/P_n}(\dot{r}). \quad (2)$$

Note that in formula (2) above that we may change $\dot{r}\cdot x_n$ to $\dot{r}t\cdot x_n$, as $t \in P_n \subseteq G_{x_n}$. Also note that we may use Haar measure to change the u part of the integral to tu. Making these changes and using the multiplicativity of characters, we get that (2) above equals

$$\int_{\dot r\in G/P_n}\int_{s\in G}\int_{t\in P_n}\int_{u\in P_n} h(s,\dot r t{\cdot}x_n)\chi_{f_n,P_n}(t)g(s^{-1}\dot r t)\overline{\chi_{f_n,P_n}(t)}\;\overline{\chi_{f_n,P_n}(u)}\;\overline{g(\dot r t u)}\cdot$$
$$\mathrm{d}\mu_{P_n}(u)\,\mathrm{d}\mu_{P_n}(t)\,\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_{G/P_n}(\dot r) =$$

(Note that the $\chi_{f_n,P_n}(t)$'s cancel)

$$\int_{\dot r\in G/P_n}\int_{s\in G}\int_{t\in P_n}\int_{u\in P_n} h(s,\dot r t\cdot x_n)g(s^{-1}\dot r t)\overline{\chi_{f_n,P_n}(u)}\;\overline{g(\dot r t u)}\cdot$$
$$\mathrm{d}\mu_{P_n}(u)\,\mathrm{d}\mu_{P_n}(t)\,\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_{G/P_n}(\dot r). \quad (3)$$

Now we combine $\dot r$ and t into a single variable $v \in G$ to get that (3) above equals

$$\int_{v\in G}\int_{s\in G}\int_{u\in P_n} h(s,v\cdot x_n)g(s^{-1}v)\overline{\chi_{f_n,P_n}(u)}\;\overline{g(vu)}\;\mathrm{d}\mu_{P_n}(u)\,\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_G(v), \quad (4)$$

and all functions in the last integral (4) are continuous with compact support, so by Lemma 67 the integral of formula (4) above converges to

$$\int_{v\in G}\int_{s\in G}\int_{u\in P} h(s,v\cdot x)g(s^{-1}v)\overline{\chi_{f,P}(u)}\;\overline{g(vu)}\;\mathrm{d}\mu_P(u)\,\mathrm{d}\mu_G(s)\,\mathrm{d}\mu_G(v) =$$
$$= \langle L_\infty(h)A(g), A(g)\rangle.$$

So by earlier remarks we conclude that $L_n \to L_\infty$.

If $P = \exp(\mathfrak{p}_x)$, for any $\mathfrak{p}_x$ a polarizing subalgebra of $\mathfrak{g}_x$ with respect to the restriction of f, we are done. Otherwise we note that $L_\infty = \mathrm{ind}_{(P,\Omega)}^{(G,\Omega)}(\chi_{f,P}, x)$, and $L = \mathrm{ind}_{(P_x,\Omega)}^{(G,\Omega)}(\chi_{f,P_x}, x)$, and the desired containment of kernels comes from Lemma 164, see also Observation 165. $\square$

Definition 169

We define a quotient space (we prove that we have an equivalence relation below) of $\mathfrak{g}^* \times \Omega$ by identifying $(f, x) \sim (f', x')$ if:

(1) There exists $s \in G$ such that $x' = s \cdot x$.

(2) For some $h \in \mathfrak{g}_{x'}^{\perp}$, we have $\mathrm{Ad}^*(s)f = f' + h$.

(2a) Equivalently, we may write (2) as:
For some $h' \in \mathfrak{g}_x^{\perp}$, we have $f = \mathrm{Ad}^*(s^{-1})f' + h'$.

Comment 170

We comment that we may also view this as

$$\mathcal{O}_{(f,x)} = \{(l, y) \in \mathfrak{g}^* \times \Omega \mid \text{for some } s \in G, \text{we have}$$

$$l = \mathrm{Ad}^*(s^{-1})f + h,\ y = s \cdot x,\ h \in \mathfrak{g}_x^{\perp}\};$$

we will use these equivalence classes in Chapter 3.

Remember, Ad^* was defined in Definition 78.

Proposition 171. *The above relation $\sim$ is an equivalence relation on $\mathfrak{g}^* \times \Omega$.*

Proof.

Reflexivity is obvious.

For symmetry, assume that we have $x' = s \cdot x$ and that $\mathrm{Ad}^*(s)f = f' + h$ as above. Then $s^{-1} \cdot (s \cdot x) = x$, and $\mathrm{Ad}^*(s^{-1})f' = f - \mathrm{Ad}^*(s^{-1})h$. Now note that $\mathrm{Ad}^*(s^{-1})h = h(s \cdot s^{-1})$, and $\mathfrak{g}_x = s^{-1}\mathfrak{g}_{s \cdot x}s$, and as $h \in \mathfrak{g}_{s \cdot x}$, we have that $\mathrm{Ad}^*(s^{-1})h \in \mathfrak{g}_x^{\perp}$ is easy. So our relation is reflexive.

For transitivity, assume that $(f, x) \sim (f', x') \sim (f'', x'')$, $x' = s_1 \cdot x$, $x'' = s_2 \cdot x'$, and

$$\mathrm{Ad}^*(s_1)f = f' + h_1, \text{ for } h_1 \in \mathfrak{g}_{x'}^{\perp},$$

and

$$\mathrm{Ad}^*(s_2)f' = f'' + h_2, \quad \text{for } h \in \mathfrak{g}_{x''}^{\perp}.$$

We have

$$\mathrm{Ad}^*(s_2 s_1)f = \mathrm{Ad}^*(s_2)\mathrm{Ad}^*(s_1)f = \mathrm{Ad}^*(s_2)(f' + h_1) = f'' + h_2 + \mathrm{Ad}^*(s_2)h_1.$$

Now we need only note that as $h_1 \in \mathfrak{g}_{x'} = \mathfrak{g}_{s_1 \cdot x}$, and by the same logic as the above we get $\mathrm{Ad}^*(s_2)h_1 \in \mathfrak{g}_{(s_2 s_1)\cdot x}^{\perp} = \mathfrak{g}_{s_1 \cdot (s_2 \cdot x)}$, and as $x'' = (s_2 s_1) \cdot x$, we are done. □

Definition 172

For X a topological space we denote the quotient topological space obtained from X by identifying points with identical closures as $X^{\sim}$. $X^{\sim}$ is called the T_0-ization of X. Note that $X^{\sim}$ is always T_0 and $X = X^{\sim}$ if X is already T_0.

Definition 173

For $C^*(G, \Omega)$ quasi-regular, define $\pi : \mathrm{Prim}(C^*(G, \Omega)) \mapsto (\Omega/G)^{\sim}$ by $\pi(P) = \mathrm{hull}(\mathrm{Res}(P))$, see Definitions 23, 24 and 26 for notation.

Lemma 174. *Suppose $C^*(G, \Omega)$ is quasi-regular:*

(1) π is continuous and surjective.

(2) If ω is a representation of G_x, then

$$hull(Res(ker(ind_{(G_x,\Omega)}^{(G,\Omega)}(\omega, M_x)))) = \overline{G \cdot x}.$$

Proof.

See [26], Lemma 4.5. □

Lemma 175. *Let $p : \Omega \mapsto (\Omega/G)^{\sim}$ be the natural map. We have that p is an open map.*

Proof.

See [18], page 221. $\square$

Observation 176

As the natural map $p : \Omega \mapsto (\Omega/G)^{\sim}$ is open (Lemma 175), it is easy to see that Ω/G is locally compact, so we use Fell's compact-open topology (section 1.4 of this thesis) on $\mathcal{K}(\,(\Omega/G)^{\sim}\,)$.

We will need a several more results before we get to the main result.

Definition 177

Let $L = (V, M) \leftrightarrow (f, x)$ be a functional-point pair in $\mathfrak{g}^* \times \Omega/\sim$, see Definition 150. Again we remind the reader of the equivalence class of (f, x) under $\sim$,

$$\mathcal{O}_L = \{(l, y) \in \mathfrak{g}^* \times \Omega \mid \text{for some } s \in G, \text{we have}$$

$$l = \mathrm{Ad}^*(s)f + h,\ y = s \cdot x,\ h \in \mathfrak{g}_{s \cdot x}^{\perp}\}\,.$$

Comment 179

Proposition 183 which follows (and we are building up to) is not necessary for this paper at this time, but may be for a future abstraction of some of the results, so we include it. It is perhaps of some independent interest. We here comment that it does not depend upon G being a nilpotent Lie group, and has been proven by Takesaki ([25], Theorem 7.2) when our orbit space Ω/G has a T_0 topology.

Comment 180

We here remind the reader of the convention that Ω is separable space, and by [20], page 146 and Theorem 16, page 125, is metrizable, hence T_4, and the Tietze Extension Theorem be applied to Ω.

Lemma 181. *Let $x \in \Omega$, G_x = the stabilizer of x in G. Let $C \subseteq G/G_x$ be compact in G/G_x. If $f \in C_c(G/G_x)$ is supported on C, we may find a sequence of continuous functions $\{f_n\}_{n=1}^{\infty} \subseteq C_0(\Omega)$ with*

$$f_n(y) \to \begin{cases} f(s) \text{ when } y = s \cdot x, & y \in C \cdot x \\ 0 & y \notin C \cdot x. \end{cases}$$

Proof.

We comment on the conclusion of this lemma. The function h on Ω defined by

$$h(y) = \begin{cases} f(s) \text{ when } y = s \cdot x, & y \in C \cdot x \\ 0 & y \notin C \cdot x \end{cases}$$

is not in general a continuous function on Ω, However, we may find a sequence of continuous functions limiting on h, so h will be Borel.

The set $C \cdot x$ is closed and compact in Ω by sequential arguments. Now simply define f' on $C{\cdot}x$ by $f'(s{\cdot}x) = f(s)$. Assume that $\{y_n\}_{n=1}^{\infty} \subseteq C{\cdot}x$, and $y_n \to y \in C{\cdot}x$. We need show that $f'(y_n) \to f'(y)$. Assume that we have a collection $\left\{\{g_n\}_{n=1}^{\infty}, g\right\} \subseteq C$ and $y_n = g_n \cdot x$ and $y = g \cdot x$, we may easily find such as C is compact in G and $C \cdot x$ is compact in Ω. As C is compact, we may assume that $g_n \to g'$, and as $g' \cdot x = g \cdot x$, and Ω is Hausdorff, we have $g' = g$. As f is continuous on C, we have that f' is continuous on $C \cdot x$ is clear by brute force. We now employ the Tietze Extension Theorem to extend to all Ω.

Now we note that $C \cdot x$ may not contain an open set. We find a sequence $\{C_n\}_{n=1}^{\infty}$ of nested compact neighborhoods with $\cap_{n=1}^{\infty} C_n = C \cdot x$. For each n, we use Urysohn's Lemma to find a continuous function g_n which is 1 on $C \cdot x$ and 0 on the closure of $\widetilde{C_n \cdot x}$. We set $f_n = g_n f'$ and we are done. □

Discussion 182

First we reference Section 1.3, Discussion 132 about induced representations; we here reproduce the relevant details as we will use them in the proof of Proposition 183 below. We assume that we are in the C^* algebra $C^*(G, \Omega)$, that y is any fixed point in Ω, and that π is a representation of $C^*(G_x, \Omega)$ acting on the Hilbert space V_π. Let ξ and η be arbitrary vectors in V_π. We define the induced representation, $L = \text{ind}_{(G_y,\Omega)}^{(G,\Omega)}(\pi)$ as follows:

We first define a $B = C_c(G_y, \Omega)$-valued inner product on the imprimitivity algebra $C_c(G, \Omega)$ by

$$\langle f, g \rangle_B (t, y) = \gamma_{G_x}(t) \int_{s \in G} \overline{f}(s, s \cdot y) g(st, s \cdot y) \mathrm{d}\mu_G(s), \tag{5}$$

where γ is a modular function, see Discussion 134 again.

The induced representation L acts on the completion of $C_c(G, \Omega) \otimes V_\pi$ with respect to the inner product defined by

$$\langle f \otimes \xi, g \otimes \eta \rangle_L = \langle \tau(\langle g, f \rangle_B) \xi, \eta \rangle_\tau \tag{6}$$

The action of $h \in C^*(G, \Omega)$ on the class of $f \otimes \xi$ is given by $(h * f) \otimes \xi$, see [18], page 204, or [26], page 340.

Proposition 183. *Let x be any fixed point in Ω, τ_1 and τ_2 be two representations of G_x such that (τ_1, M_x) and (τ_2, M_x) do not have the same* *kernel as representations of* $C^*(G_x,\Omega)$. *Defining* $L_1 = ind^{(G,\Omega)}_{(G_x,\Omega)}(\tau_1, M_x)$ *and* $L_2 = ind^{(G,\Omega)}_{(G_x,\Omega)}(\tau_2, M_x)$, *we have* $ker(L_1) \neq ker(L_2)$.

Proof.

We first define a new C^* algebra, that being $C^*(G, G/G_x)$. We observe that our orbit space $(G/G_x)/G$ for $C^*(G, G/G_x)$ *is* a T_0 space; it consists of one point. Hence by Theorem 14 $C^*(G, G/G_x)$ is Type I, and by Theorem 13 we have a canonical homeomorphism between $C^*(\widehat{G, G/}G_x)$ and $\mathrm{Prim}(C^*(G, G/G_x))$.

We identify the point $x \in \Omega$ with $x' = e \in G/G_x$ in the new orbit space. We further observe that $G_{x'} = G_x$ is clear. We use the same representations τ_1 and τ_2 of $G_{x'}$. Now define

$$L_1' = \mathrm{ind}^{(G,\Omega)}_{(G_{x'},\Omega)}(\tau_1, M_{x'})$$

and

$$L_2' = \mathrm{ind}^{(G,\Omega)}_{(G_{x'},\Omega)}(\tau_2, M_{x'}),$$

two irreducible representations of $C^*(G, G/G_x)$, and that these have different kernels is clear by the above comments and the fact that our orbit space is now T_0.

Throughout we assume that ξ_i and η_i $(i = 1, 2)$ are vectors in the space of τ_i $(i = 1, 2)$. The representation $(\tau_i, M_{x'})$ of $C^*(G_{x'}, G/G_{x'})$ we will refer to as π_i.

Now, as $\ker(L_1') \neq \ker(L_2')$, we may find an $h' \in C^*(G, G/G_{x'})$ with $h' \in \ker(L_1')$ and $h' \notin \ker(L_2')$. So we may assume that for all elementary tensors $f' \otimes \xi_1$ that $L_1'(h')(f' \otimes \xi_1) = h' * f' \otimes \xi_1 = 0$ (or more properly, the class of zero), and that for some elementary tensor $f' \otimes \xi_2$ that $L_2(h')(f' \otimes \xi_2) \neq 0$. We may further assume that f' is in the dense subset of continuous functions with compact support having the form

$$f'(s,y) = \sum_{i=1}^{l} f'_{1,i}(s) f'_{2,i}(y).$$

We may assume that each $f'_{1,i}$ in the last displayed formula is a continuous function of compact support on G, and $f'_{2,i}$ is a continuous function of compact support on G/G_x for each i, and f' and its component functions are collectively supported on $C_1 \times C_2$, where C_1 is a compact set in G and C_2 is a compact set in G/G_x.

We will refer to all inner products of the C^* algebra $C^*(G, G/G_x)$ as $\langle \cdot\ \cdot \rangle'_i$ to avoid confusion later on.

Re-iterating, we may assume by a scaling argument that

$$\langle h' * f' \otimes \xi_1, f' \otimes \xi_1 \rangle'_1 = 0,$$

and

$$\langle h' * f' \otimes \xi_2, f' \otimes \xi_2 \rangle'_2 = 1,$$

Now assume that $\{h'_n\}_{n=1}^{\infty} \subseteq C_c(G, G/G_x)$ and $h'_n \to h'$ in the topology of $C^*(G, G/G_x)$, and each h'_n has the form

$$h'_n(s,y) = \sum_{i=1}^{N_n} \phi'_{n,i}(s) \psi'_{n,i}(y),$$

and each is supported on a set $C_1^n \times C_2^n$, each C_1^n a compact set in G and each C_2^n a compact set in G/G_x.

We here note that $h'_n * f'$ (each n) is a continuous function of compact support; this is proven on pages 32-33 of [8].

We have

$$\langle h'_n * f' \otimes \xi_1, f' \otimes \xi_1 \rangle'_1 \to 0$$

$$\langle h'_n * f' \otimes \xi_2, f' \otimes \xi_2 \rangle'_2 \to 1,$$

and more explicitely, when we “untwist” formulas 5 and 6 of Discussion 182 we have

$$\langle h_n' * f' \otimes \xi_i, f' \otimes \xi_i\rangle_i' = \left\langle \pi_i(\langle f', (h_n' * f')\rangle_{B_i}')\xi_i, \xi_i\right\rangle_i' =$$

$$\left\langle \pi_i\left(\left(\gamma_{G_x}(t) \int\limits_{s\in G} \overline{f'}(s, s\cdot y)(h_n' * f')(st, s\cdot y)\mathrm{d}\mu_G(s)\right)(t,y)\right)\xi_i, \xi_i\right\rangle_i' =$$

$$\int\limits_{r\in G_x} \left(\gamma_{G_x}(r) \int\limits_{s\in G} \overline{f'}(s, s\cdot x')\right.$$

$$\left.(h_n' * f')(sr, s\cdot x')\mathrm{d}\mu_G(s)\right)\left\langle \tau_i(r)\xi_i, \xi_i\right\rangle_i' \mathrm{d}\mu_{G_x}(r) \underset{n\to\infty}{\longrightarrow}$$

$$\to \begin{cases} 0 & i = 1 \\ 1 & i = 2. \end{cases}$$

Now we return to $C^*(G,\Omega)$. All inner products in this C^* algebra we refer to by $\langle\cdot\ \cdot\rangle_i$ (no $'$'s so as to distinguish from those in $C^*(G, G/G_x)$).

Again, we have the hypothesis that $\tau_1 \ncong \tau_2$ as representations of G_x. Let

$$L_1 = \mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_1, M_x),$$

and

$$L_2 = \mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_2, M_x).$$

We define a collection of functions $\left\{\{h_{n,j}\}_{n,j=1}^{\infty}, \{f_j\}_{j=1}^{\infty}\right\}$ in $C_c(G,\Omega)$ in a special way. Define

$$f_j(s,y) = \sum_{i=1}^{l} f_{1,i}(s) f_{2,i,j}(y),$$

each $f_{1,i} = f'_{1,i}$; we do not change these functions. Now we choose $f_{2,i,j}$ to satisfy $f_{2,i,j}(s \cdot x) = f'_{2,i}(s \cdot x')$, so $f_{2,i,j}$ "behaves the same as $f'_{2,i}$" on the specific set $C_2 \cdot x$, extend to all Ω, and require

$$f_{2,i,j}(y) \underset{j\to\infty}{\longrightarrow} \begin{cases} f'_{2,i}(s \cdot x') \text{ when } y = s \cdot x, \quad s \in C_2 \\ \quad 0 \qquad \text{otherwise,} \end{cases}$$

see Lemma 181 for specifics. We note that we have

$$\lim_{j\to\infty} f_j(s,y) = \begin{matrix} f'(s, s \cdot x') \text{ when } y = s \cdot x, \quad s \in C_2 \\ 0 \qquad \text{otherwise,} \end{matrix}$$

Similarly use Lemma 181 to define a sequence $\{h_{n,j}\}_{n,j=1}^{\infty}$ with (for each fixed n)

$$h_{n,j}(s,y) \underset{j\to\infty}{\longrightarrow} \begin{cases} h'_n(s, s \cdot x') \text{ when } y = s \cdot x, \quad s \in C_2 \\ \quad 0 \qquad \text{otherwise.} \end{cases}$$

For each n and all j (fixed n) we may assume that the functions f_j and $h_{n,j}$ have support contained in the set $S_1^n \times S_2^n$, S_1^n compact in G, and S_2^n compact in Ω, see again the proof of Lemma 181. We also (for each fixed n) choose them uniformly bounded for all j.

Now we re-write our old calculations with these new functions, and follow the same steps now in the C^* algebra $C^*(G, \Omega)$:

$$\langle h_{n,j} * f_j \otimes \xi_i, f_j \otimes \xi_i \rangle_i =$$

$$\int\limits_{r\in G_x} \left(\gamma(r) \int\limits_{s\in G} \overline{f_j}(s, s \cdot x)(h_{n,j} * f_j)(sr, s \cdot x) \mathrm{d}\mu_G(s) \right) \left\langle \tau_i(r)\xi_i, \xi_i \right\rangle_i \mathrm{d}\mu_{G_x}(r). \tag{7}$$

Now we observe that the integral in s is over a compact set as f_j is of compact support, and by the way that we choose the functions $h_{n,j} * f_j$

for all j these have support contained in another compact set we denote $K_1 \times K_2$. So we must have $sr \in K_1$, and as $s \in S_1^n$ is forced already, we have $r \in ((S_1^n)^{-1} \cdot K_1) \cap G_x$ is forced, and this is compact in G_x.

Now we observe that we may apply the Bounded Convergence Theorem to the above integral, these functions are converging in j and for each n have been choosen in a bounded fashion with compact support, hence are easily bounded above by an integrable function. So for each fixed n, as we limit $j \to \infty$, formula (7) above converges to

$$\begin{cases} \langle h_n' * f' \otimes \xi_1, f' \otimes \xi_1 \rangle_1' & (i = 1) \\ \langle h_n' * f' \otimes \xi_2, f' \otimes \xi_2 \rangle_2' & (i = 2), \end{cases}$$

where we remind the reader that these inner products are in our C^* algebra $C^*(G, G/G_x)$. Furthermore, we may assume (fixed n) that for all $j \geq n$

$$\left| \langle h_{n,j} * f_j \otimes \xi_1, f_j \otimes \xi_1 \rangle_1 - \langle h_n' * f' \otimes \xi_1, f' \otimes \xi_1 \rangle_1' \right| < \frac{1}{n}$$
$$\left| \langle h_{n,j} * f_j \otimes \xi_2, f_j \otimes \xi_2 \rangle_2 - \langle h_n' * f' \otimes \xi_2, f' \otimes \xi_2 \rangle_2' \right| < \frac{1}{n},$$

where we have mixed inner products of $C^*(G, \Omega)$ and $C^*(G, G/G_x)$ in the above.

Now we may choose the diagonal sequence $\{h_{j,j}\}_{j=1}^{\infty}$ and the sequence $\{f_j\}_{j=1}^{\infty}$ in the above integrals, and for $i = 1$ the sequence converges to 0, for $i = 2$ it converges to 1. As ξ_1 was arbitrary, we now have $L_1(h_{j,j}) \to 0$ and $L_2(h_{j,j}) \not\to 0$, showing by Lemma 29 that L_1 and L_2 cannot have the same kernel, the desired result. □

Definition 184

We say that $C^*(G, \Omega)$ is *EH regular* if

(1) $C^*(G, \Omega)$ is quasi-regular,

(2) for every $P \in \text{Prim}(C^*(G, \Omega))$, there is an $x \in \Omega$ and an irreducible representation τ of G_x such that $P = \ker(\text{ind}_{(G_x, \Omega)}^{(G, \Omega)}(\tau, M_x))$.

This has been established in our case by Jon Rosenberg and Elliot Gootman, see [17].

Definition 185

Now define $\psi : \mathfrak{g}^* \times \Omega/\sim \mapsto \mathrm{Prim}(C^*(G,\Omega))$ by

$$\psi(f,x) = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, M_x)).$$

By construction, ψ factors through $\sim$, see Definition 169 for particulars on $\sim$.

Theorem 186. *ψ is a homeomorphism.*

Proof.

We have the diagram:

$$\begin{array}{ccc} \mathfrak{g}^* \times \Omega & & \\ \big\downarrow q & & \\ \mathfrak{g}^* \times \Omega/\sim & \xrightarrow{\psi} & \mathrm{Prim}(C^*(G,\Omega)) \end{array}$$

As ϕ (see Lemma 168) and the natural map q of $\mathfrak{g}^* \times \Omega$ to $\mathfrak{g}^* \times \Omega/\sim$ are continuous, we have that ψ is continuous. We have 1-1 by Theorem 7.2 of [25] and onto by construction.

Now let F be closed in $\mathfrak{g}^* \times \Omega$ and saturated with respect to $\sim$. We only need show that $\psi(F)$ is closed in $\mathrm{Prim}(C^*(G,\Omega))$.

Assume that $\{P_n\}_{n=1}^{\infty} \subseteq \psi(F)$ and $P_n \to P$. By EH regularity, we may assume that $P_n = \ker(L_n) = \ker(\mathrm{ind}_{(G_{x_n},\Omega)}^{(G,\Omega)}(\tau_{f_n,x_n}, M_{x_n}))$.

Passing to a subsequence, apply Lemma 157 to choose sequences $\{f'_n\}_{n=1}^{\infty} \subseteq \mathfrak{g}^*$ $\{y_n\}_{n=1}^{\infty} \subseteq \Omega$ with $f'_n \to f$, $y_n \to x$, and

$$\ker(L_n) = \ker(\mathrm{ind}_{(G_{y_n},\Omega)}^{(G,\Omega)}(\tau_{f'_n,y_n}, M_{y_n})),$$

and as $(f'_n, y_n) \to (f, x)$ in $\mathfrak{g}^* \times \Omega$, and F is saturated and closed, we are done. $\square$

Comment 187

As our orbit space Ω/G has a T_0 topology, we have established a homeomorphism to $\widehat{C^*(G, \Omega)}$, see Theorem 16.

Chapter 3

Traces of irreducible representations of $C^*(G,\Omega)$

Discussion 188

In this chapter we give a character theory for $C^*(G,\Omega)$ with G nilpotent Lie analogous to Kirillov's character theory for G, which is given below.

Section 3.1

Preliminaries

Discussion 189

In this section we give some general information about trace class operators, which we will use without reference. For further on trace-class operators, consult [24]. For $A \in \mathcal{B}(H)$, the bounded operators on a Hilbert space H, A is said to be *Hilbert-Schmidt* if for some orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$, and for the norm $\|\cdot\|_2$,

$$\|A\|_2^2 = \sum_{n=1}^{\infty} \|A\xi_n\|^2 = \sum_{n,m=1}^{\infty} |\langle A\xi_n, \xi_m\rangle|^2 < \infty$$

for some orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$. Then A is a compact operator, and the Hilbert-Schmidt norm does not depend on the basis, and these operators are a two-sided ideal in $\mathcal{B}(H)$. If $A = U|A|$ is the polar decomposition of A, then $\|A\|_2^2 = \sum_{n=1}^{\infty} |\lambda_n|^2$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of $|A|$, counted with multiplicity.

An operator A is *trace class* or *nuclear* if $|A|^{\frac{1}{2}}$ is Hilbert Schmidt, i.e., for the norm $\|\cdot\|_1$ (below), we have

$$\|A\|_1 = \sum_{n=1}^{\infty} \langle |A|\xi_n, \xi_n\rangle = \sum_{n=1}^{\infty} \|\,|A|^{\frac{1}{2}}\xi_n\|^2 = \|\,|A|^{\frac{1}{2}}\|_2^2 < \infty$$

for any orthonormal basis. Obviously, $\|A\|_1 < \infty$ implies $\|A\|_2 < \infty$. We have $\|A\|_1 = \sum_{n=1}^{\infty} \lambda_n$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of $|A|$. For A trace class, we define the *trace* of A,

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} \langle A(\xi_n), \xi_n\rangle,$$

and this is absolutely convergent and independent of the basis $\{\xi_n\}_{n=1}^{\infty}$ of H. Conversely, for $A \in \mathcal{B}(H)$,

A is trace class $\iff \sum_{n=1}^{\infty} \langle A\xi_n, \xi_n\rangle$ exists for all orthonormal bases (in which case all sums have the same value.)

As these sums are independant of rearrangements, they are absolutely convergent. Also, trace class operators form a $*$-ideal in $\mathcal{B}(H)$ and are compact operators.

Assume that a representation L of $C^*(G, \Omega)$ is modeled on $L^2(\mathbb{R}^k)$. In this paper, our operators will often have kernels K in $\mathbb{R}^k \times \mathbb{R}^k$, that is

$$(Af)(s) = \int_{t\in\mathbb{R}^k} K(s,t) f(t) \mathrm{d}t.$$

Here A is Hilbert-Schmidt $\iff K \in L^2(\mathbb{R}^{2k})$, and also

$$\|A\|_2^2 = \int |K(s,t)|^2 \mathrm{d}s\, \mathrm{d}t.$$

To determine nuclearity from the nature of the kernel is much more difficult. Some results are known, to wit:

Theorem 190. *Let A be an integral operator with kernel $K(x,y) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, the Schwartz space on $\mathbb{R}^n \times \mathbb{R}^n$ (see below).*

(a) A is a trace class operator.

(b) $Tr(A) = \int\limits_{x\in\mathbb{R}^n} K(x,x)\,dx$

Proof.

See [3], Theorem A.3.9, page 250. □

Section 3.2

Schwartz functions on G.

Discussion 191

In this section we describe the Schwartz functions on a nilpotent Lie group G; these are important in the character theory of such groups.

Discussion 192

On $\mathbb{R}^n$, the Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ are those C^∞ functions f such that

$$\|x^\beta \mathrm{D}^\alpha f\|_\infty < \infty,\ x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n},\ \ \mathrm{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$. The natural (metrizable) topology of $\mathcal{S}(\mathbb{R}^n)$ is determined by these seminorms. If we denote the polynomial coefficient differential operators on $\mathbb{R}^n$ by $\mathcal{P}(\mathbb{R}^n)$, it it clear from the above that $f \in \mathcal{S}(\mathbb{R}^n) \iff \|L(f)\|_\infty < \infty$ for all $L \in \mathcal{P}(\mathbb{R}^n)$.

Lemma 193. *Let* $p : \mathbb{R}^n \mapsto \mathbb{R}^n$ *be a polynomial diffeomorphism (*p *and* p^{-1} *polynomial maps).*

(a) The map $f \mapsto f \circ p$ *is a topological isomorphism of* $\mathcal{S}(\mathbb{R}^n)$

(b) The map p^*L*, defined by* $p^*L(f) = L(f \circ p) \circ p^{-1}$*, is an algebraic isomorphism of* $\mathcal{P}(\mathbb{R}^n)$

Proof.

See [3], Lemma A.2.1, page 235. □

Definition 194

On G we define $\mathcal{S}(G)$, the Schwartz functions on G, to be the functions on G that correspond to $\mathcal{P}(\mathbb{R}^n)$ under a polynomial coordinate map $\psi : \mathbb{R}^n \mapsto G$. By the last lemma, this does not depend upon the choice of ψ.

We need several more results about Schwartz space.

Definition 195

A *tempered distribution* on Schwartz space is a linear functional that is continuous in the topology of Schwartz space.

Proposition 196. *Let* T *be a tempered distribution on* $\mathcal{S}(G)$*. There exists some constant* C*, and multi-indices* α *and* β *such that*

$$|T(f)| \leq C \cdot \left| x^\beta D^\alpha f(x) \right| \ \textit{for all } f \in C^\infty(G).$$

Proof.

See [27], Corollary 1, page 43. We comment that [27]'s version of this is slightly more general. □

Proposition 197. *The Fourier transform of a Schwartz function is also Schwartz.*

Proof.

See [27], Proposition 4 on page 146. □

Section 3.3

Traces on $C^*(G)$

Discussion 198

In this section we describe some operators on G which are analogues of classical trace operators on finite and compact groups.

For references on the following discussion, see [3], from which it is abstacted.

Discussion 199

For π a unitary infinite-dimensional irreducible representation of a nilpotent Lie group G, the operators $\pi(x)$ have no trace. However, for any $\pi \in \widehat{G}$, there is a tempered distribution θ_π on G that plays the role of the classical trace character $\theta_\pi(g) = \mathrm{Tr}(\pi(g))$ for finite and compact groups. For $\pi \in \mathcal{S}(G)$, the Schwartz functions on G, the operator

$$\pi(\phi)(\xi) = \int\limits_{s \in G} \phi(s)\pi(s)\,\xi\,\mathrm{d}\mu_G(s) \quad (\xi \in H_\pi)$$

turns to be trace class. To wit:

Theorem 200. *Let $\pi = \pi_l$ be an irreducible representation of a nilpotent Lie group, let $\mathfrak{m}$ be a polarization for l, and model π in $L^2(\mathbb{R}^k)$ using any weak Malcev basis through $\mathfrak{m}$. If $\phi \in \mathcal{S}(G)$, then $\pi(\phi)$ is trace class and*

$$\pi_\phi f(s) = \int_{\mathbb{R}^k} K_\phi(s,t) f(t)\, dt, \text{ for all } f \in L^2(\mathbb{R}^k)$$

where $K_\phi \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$. Furthermore,

$$\theta_\pi(\phi) = Tr(\pi(\phi)) = \int_{\mathbb{R}^k} K_\phi(s,s)\, ds \text{ (absolutely convergent)}$$

and the functional θ_π is a tempered distribution on $\mathcal{S}(G)$.

Proof. See [3], Theorem 4.2.1, page 133. □

Definition 201

Now we know the properties of the kernel K_ϕ, which is unique, it is easy to obtain explicit formulas for it once an $l \in \mathfrak{g}^*$, a polarization $\mathfrak{m}$, and a weak Malcev basis through $\mathfrak{m}$ are specified. If $\{X_1, ..., X_n\}$ is the basis, let $p = n - k = \dim(\mathfrak{m})$. Define polynomial maps $\gamma : \mathbb{R}^n \mapsto G,\ \alpha : \mathbb{R}^p \mapsto M,\ \beta : \mathbb{R}^k \mapsto G$ by

$$\gamma(s,t) = \exp(s_1 X_1) \cdots \exp(s_p X_p) \cdot \exp(t_1 X_{p+1}) \cdots \exp(t_k X_n)$$

$$\alpha(s) = \gamma(s,0), \quad \beta(t) = \gamma(0,t)$$

and let $d\mu_G,\ d\mu_M,\ d\mu_{G/M}$ be the invariant measures on $G, M, G/M$ determined by Lebesgue measures $ds\, dt, ds,\ dt$ as in Theorem Theorems 46, 47 and 48.

Proposition 202. *If we take the standard basis realization of $\pi = \pi_l$ in $L^2(\mathbb{R}^k)$ relative to the given Malcev basis, the kernel K_ϕ has the form*

$$K_\phi(s,t) = \int_{m \in M} \chi_l(m) \phi\left(\beta(t) \cdot m \cdot \beta(s)^{-1}\right) d\mu_M(m) \text{ (absolutely convergent)}$$

where $\chi_l(exp(Y)) = e^{i\cdot l(Y)}$ *for* $Y \in \mathfrak{m}$ *and* β *is the map defined above.*

Proof.

We will prove this in greater generality in Proposition 209 later on, also see [3], Proposition 4.2.2 for its form when we are in a group case. We comment that our kernel looks different from [3]'s as [3] uses right actions and we are using left. □

Discussion 203

We comment that the following discussion is abstracted from [3], pp. 137-8.

We now use the above formula to describe $\mathrm{Tr}(\pi(\phi))$ in terms of integrals over coadjoint orbits (see Definition 80) in $\mathfrak{g}^*$. Given a Euclidean measure $\mathrm{d}X$ on $\mathfrak{g}$, we normalize measures on $\mathfrak{g}$ and $\mathfrak{g}^*$ so that Fourier inversion holds, and we define the *Euclidean Fourier transforms* $\widehat{f}$ (resp. $\mathcal{F}f$) of functions on G (resp. $\mathfrak{g}$) to be

$$\left.\begin{aligned} \widehat{f}(l) &= \int\limits_{X\in\mathfrak{g}} e^{i\cdot l(X)} f(\exp(X))\ \mathrm{d}X, \\ \mathcal{F}f(l) &= \int\limits_{X\in\mathfrak{g}} e^{i\cdot l(X)} f(X)\ \mathrm{d}X, \end{aligned}\right\} \text{for all } l \in \mathfrak{g}^*.$$

Each coadjoint orbit $\mathcal{O}_l = (\mathrm{Ad}^*(G))l$ has an invariant measure μ that is unique up to scalar multiple as $\mathcal{O}_l \cong R_l/G$, where

$$R_l = \mathrm{Stab}_G(l) = \{x \in G \mid \mathrm{Ad}^*(x)l = l\}.$$

Theorem 204. *If* π *is an irreducible representation of a nilpotent Lie group, corresponding to the co-adjoint orbit* $\mathcal{O}_l = Ad^*(G)l \subseteq \mathfrak{g}^*$*, there is a unique choice of invariant measure* μ *on* $\mathcal{O}_l$ *such that*

$$Tr(\pi(\phi)) = \int\limits_{l\in\mathfrak{g}^*} \widehat{\phi}(l)\mu_l(dl) \text{ for all } \phi \in \mathcal{S}(G),$$

with the integral absolutely convergent.

Proof.

See [3], Theorem 4.2.4, page 138.

Comment 205

We observe that it is included in the proof of the above that the the measure is $c \cdot \mathrm{d}\dot{g}$, where $\mathrm{d}\dot{g}$ is invariant measure on R_l/G ($R_l = \exp(\mathfrak{r}_l)$ and $\mathfrak{r}_l$ = a polarizing subalgebra of $\mathfrak{g}$ with respect to the action of l), and c is determined by the next theorem.

Theorem 206. *Let G be a nilpotent Lie group, let $\pi = \pi_l$ be an irreducible representation (for $l \in \mathfrak{g}^*$). Let $\{U_1, \ldots, U_r, X_1, \ldots, X_{2k}\}$ be any weak Malcev basis through $\mathfrak{r}_l$, and let $\gamma : \mathbb{R}^{2k} \mapsto R_l\backslash G$ be the Malcev coordinate map. If B is the $2k \times 2k$ matrix with entries $B_l(X_i, X_j) = l([X_i, X_j])$, then Euclidean measure $dx = dx_1 \wedge \cdots \wedge dx_{2k}$ on $\mathbb{R}^{2k}$ determines a particular invariant measure on $R_l\backslash G$ such that*

$$\begin{aligned} Tr(\pi(\phi)) &= |det\ B|^{1/2} \int\limits_{x \in \mathbb{R}^{2k}} \widehat{\phi}(l \cdot \gamma(x))\, dx \\ &= |det\ B|^{1/2} \int\limits_{g \in R_l\backslash G} \widehat{\phi}(l \cdot g)\, d\dot{g} \end{aligned}$$

for any Schwartz function on G.

Proof.

See [3], Theorem 4.2.5, page 141.

Traces of irreducible representations of $C^*(G,\Omega)$

Discussion 207

In this section we characterize some operators coming from $C^*(G,\Omega)$ which are trace class. First define $\mathcal{S}(G)$ to be the set of functions on G which are Schwartz, see Discussion 192. We assume that G is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, these are fixed for the remainder of this section. First we establish a small inequality that we will need in one of our proofs.

Lemma 208. *(Peetre's Inequality) Let s be real and $x, y \in \mathbb{R}^m$. Then*

$$(1+|x+y|)^s \leq (1+|y|)^s(1+|x|)^{|s|}.$$

Proof.

See [14], page 10. $\square$

Proposition 209. *Let $\phi \in \mathcal{S}(G)$, $\psi \in C_0(\Omega)$. Also let $L = (V, M) \leftrightarrow (f, x)$, let $k = dim(\mathfrak{g}/\mathfrak{p})$, where $\mathfrak{p}$ is polarizing for the action of f in $\mathfrak{g}_x$. Then if L is modeled on $L^2(\mathbb{R}^k)$, then for $\psi \in C_0(\Omega)$ and $\phi \in \mathcal{S}(G)$, then $L(\phi \cdot \psi)$ has as a kernel K, where*

$$K(r,s) = \psi(exp(r)\cdot x)\int\limits_{t\in P} \phi(\beta(r)t\beta(s)^{-1})e^{i\cdot f(\log(t))}d\mu_{\mathfrak{p}}(t)$$

Proof.

See Comment 142 for how L is acting. For $\phi \in \mathcal{S}(G)$, $\psi \in C_0(\Omega)$ such that $\psi(\cdot\, x)|_{G/G_x} \in \mathcal{S}(G/G_x)$, $h \in L^2(\mathbb{R}^k)$ we have

$$(L(\phi\cdot\psi)h)(\exp(r)) = \psi(\exp(r)\cdot x)\int\limits_{s\in\mathfrak{g}} \phi(\exp(s))h(\exp(s)^{-1}\exp(r))\mathrm{d}\mu_{\mathfrak{g}}(s) =$$

(letting $s \to s^{-1}$, our measure doesn't change by Lemma 52)

$$\psi(\exp(r)\cdot x)\int\limits_{s\in\mathfrak{g}}\phi(\exp(s^{-1}))h(\exp(s)\exp(r))\mathrm{d}\mu_{\mathfrak{g}}(s) =$$

(letting $s \to sr^{-1}$)

$$\psi(\exp(r)\cdot x)\int\limits_{s\in\mathfrak{g}}\phi(\exp(r)(\exp(s^{-1})))h(\exp(s))\mathrm{d}\mu_{\mathfrak{g}} =$$

(splitting $\mathfrak{g}$ into $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}$)

$$\psi(\exp(r)\cdot x)\int\limits_{s\in\mathfrak{g}/\mathfrak{p}}\int\limits_{t\in\mathfrak{p}}\phi(\exp(r)(\exp(t^{-1})\exp(s^{-1})))h(\exp(s)\exp(t))\mathrm{d}\mu_{\mathfrak{g}} =$$

$$\psi(\exp(r)\cdot x)\int\limits_{s\in\mathfrak{g}/\mathfrak{p}}\int\limits_{t\in\mathfrak{p}}\phi(\exp(r)(\exp(t^{-1})\exp(s^{-1})))e^{i\cdot f(t)}h(\exp(s))\mathrm{d}\mu_{\mathfrak{g}}$$

We now let

$$K(r,s) = \psi(\exp(r)\cdot x)\int\limits_{t\in\mathfrak{p}}\phi(\beta(r)\exp(t)\beta(s)^{-1})e^{i\cdot f(t)}\mathrm{d}\mu_{\mathfrak{p}}(t),$$

and the result is clear. □

Definition 210

Let H be a Hilbert space, we assume that $H \cong L^2(\mathbb{R}^k)$. For a function $\psi \in C_0(\mathbb{R}^k)$, the multiplication operator on H defined by ψ we denote by M_ψ.

Comment 211

The next theorem is an important one, enabling us to prove our first trace theorem.

Lemma 212. *Let ρ be an irreducible representation of a subgroup H of G, corresponding to the restriction of a functional $f \in \mathfrak{g}^*$ to $\mathfrak{h}$. Defining $\pi = ind_H^G(\rho)$, we have that the the operator $T_{\psi,\phi} = M_\psi \cdot \pi(\phi)$ is trace class with a Schwartz kernel when $\psi \in \mathcal{S}(G/H)$ and $\phi \in \mathcal{S}(G)$. Also, for fixed ψ, $T_{\psi,\phi}$ is tempered in ϕ.*

Proof.

We do this by induction on the codimension of H in G. If $\text{codim}(H) = 0$, this is true by Theorem 200.

Assume that the lemma is true for $\text{codim}(H) = n$. We show it for $\text{codim}(H) = n + 1$.

By Theorem 42 we find a subgroup $G_1 \subseteq G$ with $\text{codim}(G_1) = 1$ and $H \subseteq G_1$. We note G_1 is normal in G by Corollary 51, and that H is codimension n in G_1. We assume that the Lie algebra of G_1 is $\mathfrak{g}_1$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \big(\mathbb{R} - \text{span}\{X\}\big)$; we have a smooth cross section for G/G_1 by $\exp(\mathbb{R} \cdot X) \cong \mathbb{R}$.

Define $\tau = \text{ind}_H^{G_1}(\rho)$, acting on the Hilbert space H_τ.

We assume that $\pi = \text{ind}_H^G(\rho) \cong \text{ind}_{G_1}^G(\tau)$ acts on the Hilbert space $H_\pi = L^2(\mathbb{R}) \otimes H_\tau$.

For $f \in C^\infty(H_\pi)$, and an element $a \in G/G_1$, by $f(a)$ we mean the element of H_τ corresponding to $f(a)$, for $z \in G_1$, we will refer to $f(a)(z)$ as the value of the H_τ-valued function $f(a)$ in H_τ at the point z in G_1. Let $\phi_1 \in \mathcal{S}(G/G_1)$, $\phi_2 \in \mathcal{S}(G_1)$, $\psi_1 \in \mathcal{S}(G/G_1)$, and $\psi_2 \in \mathcal{S}(G_1/H)$. We here note that sums of elements of the form $\phi_1 \cdot \phi_2$ are dense in $\mathcal{S}(G)$, and that sums of the form $\psi_1 \cdot \psi_2$ are dense in $\mathcal{S}(G/H)$.

Let $r \in G/G_1$. Let $f \in C^\infty(H_\pi)$. We have

$$(M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)f)(r) = M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)\pi(t)f(r)\mathrm{d}s\,\mathrm{d}t =$$

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)f(t^{-1}r)\mathrm{d}s\,\mathrm{d}t =$$

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)f(rr^{-1}t^{-1}r)\mathrm{d}s\,\mathrm{d}t =$$

(Note that $r^{-1}tr \in G_1$ by normality)

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)\tau(r^{-1}tr)f(r)\mathrm{d}s\,\mathrm{d}t =$$

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\tau(r^{-1}sts^{-1}r)f(s^{-1}r)\mathrm{d}t\,\mathrm{d}s =$$

(Let $t \to s^{-1}rtr^{-1}s$ in the inner integral)

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\phi_1(s)\int\limits_{t\in G_1}\phi_2(s^{-1}rtr^{-1}s)\tau(t)f(s^{-1}r)\mathrm{d}t\,\mathrm{d}s =$$

(Let $s^{-1}r \to a$, we note that $a \in G/G_1$)

$$M_{\psi_1}\int\limits_{a\in G/G_1}\phi_1(ra^{-1})\left[M_{\psi_2}\int\limits_{t\in G_1}\phi_2(ata^{-1})\tau(t)\mathrm{d}t\right]f(a)\mathrm{d}a. \tag{8}$$

We note that for fixed a, by induction hypothesis we have that the operator in the brackets of (8) above is a trace class operator on H_τ, and for fixed ψ_2, is tempered in ϕ. We now employ Lemma 50 to get that for some selection of polynomials $\{P_i\}_{i=1}^n$ and at a fixed point $y \in G_1$ the inner integral of formula (8) equals

$$\left(M_{\psi_2} \int\limits_{t \in G_1} \phi_2 \left(P_1(a,t), \dots P_n(a,t)\right) \tau(t) f(a) \mathrm{d}t \right)(y) = \int\limits_{z \in G_1} k^a_{\phi_2}(z,y) f(a)(z) \mathrm{d}z.$$

We note that we may find the integral kernel $k^a_{\phi_2}(z,y)$ by inductive hypothesis, and it may be chosen Schwartz in y and z by inductive hypothesis.

We now recall that for fixed a that $\mathrm{Tr}(k^a_{\phi_2})$ is tempered in ϕ_2 by inductive hypothesis. By Proposition 196 we know that there exists a seminorm ρ on G_1 such and a constant C such that that $|\mathrm{Tr}(k^a_{\phi_2}))| \leq C \cdot \rho(\phi_2^a)$, where ϕ_2^a is defined as obvious.

Assume that we have multi-indices $\alpha, \beta \in \mathbb{Z}^n$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = \beta_1, \dots, \beta_n$, and we have for the seminorm ρ:

$$|\rho(\phi^a)| =$$

$$\sup_{w \in \mathbb{R}^n} \left\{ \left| \, w_1^{\beta_1} \cdots w_n^{\beta_n} \cdot \frac{\partial^{|\alpha|}}{\partial w_1^{\alpha_1} \cdots \partial w_n^{\alpha_{i-1}}} \, \phi_2 \left(P_1(a,w), \dots, P_n(a,w)\right) \right| \, \right\} \leq$$

$$\leq |Q(a)|,$$

by Lemma 49 and the n dimensional chain rule, where Q is a polynomial.

Putting this together, $\mathrm{Tr}(k^a_{\phi_2}) = \int_{y \in G_1} k^a_{\phi_2}(y,y) \mathrm{d}y$ (See Theorem 190) grows no faster than polynomial in a, and $k^a_{\phi_2}$ grows no faster that polynomial in a.

Now note that for a in a bounded set, we have $\phi_2(ata^{-1})$ is easily bounded by a L^1 function. Now note that by Proposition 209 that when $\mathfrak{p}$ is a polarizing subalgebra of $\mathfrak{g}_x$ with respect to the restriction of f, we have

$$k^a_{\phi_2}(r,s) = \psi_2(\exp(r) \cdot x) \int\limits_{t \in P} \phi_2(\beta(r) a t a^{-1} \beta(s)^{-1}) e^{i \cdot f(\log(t))} \mathrm{d}\mu_{\mathfrak{p}}(t),$$

and to differentiate in a under the integral sign is easy, and infinite differentiabilty of $k^a_{\phi_2}$ in a follows.

Now we choose an smooth splitting of G into G/G_1 and G_1 by the projections p and q, $p : G \mapsto G/G_1$, and $q : G \mapsto G_1$.

Now for our functions ϕ_1, ϕ_2, ψ_1 and ψ_2 we define the integral kernel K by

$$K(x, y) = M_{\psi_1}\big(p(y)\big) \cdot \phi_1\big(p(y)p(x^{-1})\big) \cdot k^{p(y)}_{\phi_2}\big(q(x), q(y)\big).$$

We note that integrated against $f \in H_\pi$, this gives us $M_{\psi_1 \cdot \psi_2}\pi(\phi_1 \cdot \phi_2)(f)$, viewing K as an operator:

$$(Kf)(y) = \int\limits_{x \in G} K(x, y) f(x) \mathrm{d}x =$$

$$M_{\psi_1}(p(y)) \int\limits_{x \in G} \phi_1(p(y)p(x^{-1})) k^{p(y)}_{\phi_2}(q(x), q(y)) f(x) \mathrm{d}x =$$

$$M_{\psi_1}(p(y)) \int\limits_{a \in G/G_1} \int\limits_{z \in G_1} \phi_1(p(y)a^{-1}) k^a_{\phi_2}(z, q(y)) f(a)(z) \mathrm{d}z \, \mathrm{d}a =$$

$$(M_{\psi_1 \cdot \psi_2}\pi(\phi_1 \cdot \phi_2) f)(y).$$

We note that $k^a_{\phi_2}$ is Schwartz on G_1 and its integral and the integrals of its derivitives grow no faster than polynomial in a; we will henceforth ignore the parameter a.

We note that $G/G_1 \cong \mathbb{R}$, so we henceforth treat composition on G/G_1 as addition on $\mathbb{R}$. Now we note that ψ_1 and ϕ_1 are both Schwartz on G/G_1, and part of the integral kernal looks like $M_{\psi_1}\big(p(y)\big) \cdot \phi_1\big(p(y)p(x^{-1})\big)$ and the other part is Schwartz already.

So our problem reduces to: for ψ and ϕ both Schwartz on $\mathbb{R}$, we need to verify that $\psi(r)\phi(r + s)$ is Schwartz on $\mathbb{R} \times \mathbb{R}$.

Note that as ψ and ϕ are Schwartz that

$$|\psi(r)\phi(r+s)| \leq C_{m,n} \cdot (1+|s|)^{-m}(1+|r+s|)^{-n},$$

where C depends on the positive integers m and n. By Peetre's inequality (Lemma 208), we have

$$|\psi(r)\phi(r+s)| \leq C_{m,n} \cdot (1+|s|)^{-m}(1+|r+s|)^{-n}$$
$$\leq C_{m,n}(1+|s|)^{-m}(1+|r|)^{-n}(1+|s|)^{|-n|} = (1+|s|)^{-m+n}(1+|r|)^{-n}.$$

By varying $C_{m,n}$, we may choose m as large as we like, verifying that growth conditions of Schwartz functions. The similar properties of the derivitives are clear, and we have that $\psi(r)\phi(r+s)$ is Schwartz on $\mathbb{R}^2$.

So our entire integral kernel is Schwartz, and the operator $M_{\psi_1 \cdot \psi_2}\pi(\phi_1 \cdot \phi_2)$ is trace class by Theorem 190. We also observe that the kernel K is clearly tempered in $\phi_1 \in \mathcal{S}(G/G_1)$. The final result follows by induction and density arguments. □

Convention 213

We must here make the convention that Ω/G is a T_0 space, and (G, Ω) is Polish, see Proposition 18, thus we have that for any $x \in \Omega$ we have $G \cdot x \cong G/G_x$. We here comment that when Ω/G is non-T_0 then there exists an irreducible representation L of $C^*(G, \Omega)$ such that for a any non-zero element of $C^*(G, \Omega)$ we have $L(a)$ is not a compact operator, hence $L(a)$ cannot be trace-class, see again Discussion 189.

Definition 214

For any orbit $G \cdot x$, define

$$\mathcal{A}_{G\cdot x} = \mathbb{R} - \text{span}\big\{\phi \cdot \psi \mid \phi \in \mathcal{S}(G),\ \psi \in C_0(\Omega) \text{ and } \psi(\cdot\ x)|_{G/G_x}$$
$$\text{is in } \mathcal{S}(G/G_x)\big\}.$$

Theorem 215. *Let $L = (V, M)$ be the irreducible representation of $C^*(G, \Omega)$ associated to (f, x). The representation L is trace class on $\mathcal{A}_{G \cdot x}$, and for fixed ψ, is tempered in ϕ.*

Proof.

Note that $L(\psi \cdot \phi) = M_\psi(\cdot\ x) \cdot V(\phi)$ and appeal to Lemma 212 just proven. $\square$

Definition 216

Remember Definition 201, where we defined maps $\alpha,\ \beta,\ \gamma$ from $\mathfrak{g}$ to G. Assume that we have a functional $f \in \mathfrak{g}^*$, and that $\mathfrak{g}_x$ is the Lie algebra of G_x for $x \in \Omega$, and that $\mathfrak{p}$ is a polarizing subalgebra for the restriction of f to $\mathfrak{g}_x$. We assume that $\dim(\mathfrak{g}_x) = l$. We define a new map $\delta : \mathbb{R}^l \mapsto G_x$ by $\delta(s) = \gamma(s_1, \ldots, s_l, 0)$.

Theorem 217. *Let L be a representation of $C^*(G, \Omega)$ corresponding to the functional-point pair (f, x), modeled on $L^2(\mathbb{R}^k)$ $(k = dim(\mathfrak{g}/\mathfrak{p}))$, where $\mathfrak{p} =$ a polarizing subalgebra of $\mathfrak{g}_x$ with respect to the restriction of f to $\mathfrak{g}_x$. Let $\mathcal{O}_L$ be the equivalence class of (f, x), specifically,*

$$\mathcal{O}_{(f,x)} = \{(l, y) \in \mathfrak{g}^* \times \Omega \mid \textit{for some } s \in G, \textit{we have}$$

$$l = Ad^*(s^{-1})f + h,\ y = s \cdot x,\ h \in \mathfrak{g}_x^\perp\}.$$

see Definition 169 and comment 170. Let p, q be the natural projections from $\mathcal{O}_L$ to $\mathfrak{g}^*$ and Ω, respectively. Let $\phi \in \mathcal{S}(G)$, $\psi(\cdot\ x) \in \mathcal{S}(G/G_x)$. We have

$$Tr(L(\phi \cdot \psi)) = \int\limits_{z \in \mathcal{O}_L} \psi(q(z))\widehat{\phi}(p(z))dz$$

for a particular choice of G-invariant measure dz on $\mathcal{O}_L$.

Proof.

We closely mimic the proof of Theorem 4.2.4 on pp. 138-41 of [3].

Assume that $\mathfrak{p}$ has dimension p, so $n = \dim(\mathfrak{g}) = p + k$. Also, as G_x is fixed, denote it by S, and its Lie algebra will be denoted by $\mathfrak{s}$.

We give $\mathfrak{g}$ a standard basis realization on $L^2(\mathbb{R}^k)$. By Proposition 209 above, and Theorem 190, we have

$$\mathrm{Tr}(L) = \int\limits_{s\in\mathfrak{g}/\mathfrak{p}} K(s,s)\mathrm{d}s =$$

$$\int\limits_{s\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(s)\cdot x) \int\limits_{t\in\mathfrak{p}} \phi(\beta(s)\exp(t)\beta(s)^{-1})e^{i\cdot f(t)}\mathrm{d}\mu_{\mathfrak{p}}(t)\,\mathrm{d}\mu_{\mathfrak{g}/\mathfrak{p}}(s). \tag{9}$$

Let $\mathfrak{z}$ be a subspace complementary to $\mathfrak{p}$ in $\mathfrak{g}$. It is easy to take $\mathfrak{z} = \mathbb{R}$-span of the last k basis vectors of our Malcev basis through $\mathfrak{p}$. We have an additive splitting $H + X \in \mathfrak{p} \oplus \mathfrak{z}$ for each element in $\mathfrak{g}$. Let $\mathrm{d}X$, $\mathrm{d}H$ be arbitrarily assigned Euclidean measures on $\mathfrak{z}$, $\mathfrak{p}$; then we have that $\mathrm{d}H\,\mathrm{d}X$ is a Euclidean measure on $\mathfrak{g}$, which we use to define the above integrals.

For $\phi \in \mathcal{S}(G),\ u \in \mathbb{R}^k \cong \mathfrak{z} \cong \mathfrak{s}/\mathfrak{p}$, we define

$$\phi_u(H,X) = \phi(\beta(u)\exp(H+X)\beta(u)^{-1}).$$

For each fixed u, this is a Schwartz function on $\mathbb{R}^p \times \mathbb{R}^k \cong \mathfrak{p} \times \mathfrak{z}$. Viewing $\mathfrak{z}^* \cong \mathfrak{p}^\perp$, define the Fourier transform

$$\mathcal{F}\phi(f^\perp) = \int\limits_{X\in\mathfrak{z}} \phi(X)e^{i\cdot f^\perp(X)}\mathrm{d}f^\perp \quad \text{for } \phi \in \mathcal{S}(Z),\ f^\perp \in \mathfrak{p}^\perp.$$

When the measures are suitably normalized, this is an L^2 isometry with inverse

$$(\mathcal{F}^{-1}\phi)(X) = \int\limits_{f^\perp\in\mathfrak{p}^\perp} \phi(f^\perp)e^{-i\cdot f^\perp(X)}\mathrm{d}f^\perp,$$

taking the dual Euclidean measure $\mathrm{d}l^{\perp}$ on $\mathfrak{p}^{\perp}$. We have

$$\phi_u(H,0) = (\mathcal{F}^{-1}\mathcal{F}\phi_u)(H,0) = \int\limits_{f^{\perp}\in\mathfrak{z}^*} e^{-i\cdot f^{\perp}(0)}\mathcal{F}\phi_u(H,f^{\perp})\mathrm{d}f^{\perp} =$$

$$\int\limits_{f^{\perp}\in\mathfrak{p}^{\perp}} \int\limits_{X\in\mathfrak{z}} \phi_u(H,X)e^{i\cdot f^{\perp}(X)}\mathrm{d}X\,\mathrm{d}f^{\perp}. \tag{10}$$

We want to insert equation 9 into equation 10 and interchange some integrals, this will be done introducing and removing an ad hoc function which enables us to use Fubini's Theorem. Let $\{w_j\}_{j=1}^{\infty}$ be a collection of functions on $\mathfrak{p}^{\perp}$ with $0 \le w_j(f^{\perp}) \le 1$, and $w_j \uparrow 1$ uniformly on compacta in $\mathfrak{p}^{\perp}$ and $\int\limits_{f^{\perp}\in\mathfrak{p}^{\perp}} w_j(f^{\perp})\mathrm{d}f^{\perp} < \infty$ for all j. As $\mathcal{F}\phi_u(H,f^{\perp})$ is Schwartz in both variables (remember that the Fourier transform of a Schwartz function is also Schwartz, see Proposition 197), by dominated convergence we have

$$\mathrm{Tr}(L) = \int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x) \int\limits_{H\in\mathfrak{p}} e^{i\cdot f(H)}\phi_u(H,0)\mathrm{d}H\,\mathrm{d}u =$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x) \int\limits_{H\in\mathfrak{p}} \left[\lim_{j\to\infty} \int\limits_{f^{\perp}\in\mathfrak{p}^{\perp}} \mathcal{F}\phi_u(H,f^{\perp})w_j(f^{\perp})\mathrm{d}f^{\perp}\right] e^{i\cdot f(H)}\mathrm{d}H\,\mathrm{d}u =$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\Bigg[\lim_{j\to\infty} \int\limits_{H\in\mathfrak{p}} \int\limits_{f^{\perp}\in\mathfrak{p}^{\perp}} \int\limits_{X\in\mathfrak{z}} e^{i\cdot f(H)}e^{i\cdot f^{\perp}(X)}\phi_u(H,X)w_j(f^{\perp})\cdot$$

$$\mathrm{d}X\,\mathrm{d}f^{\perp}\,\mathrm{d}H\Bigg]\mathrm{d}u.$$

Fubini's Theorem may be applied to the innermost triple integral, when we re-arrange, we get

$$\mathrm{Tr}(L) =$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\Bigg[\lim_{j\to\infty} \int\limits_{f^{\perp}\in\mathfrak{p}^{\perp}} w_j(f^{\perp}) \int\limits_{\{H,X\}\in\mathfrak{p}\oplus\mathfrak{z}} e^{i\cdot\left(f(H)+f^{\perp}(X)\right)}\phi_u(H,X)$$

$$\mathrm{d}X\,\mathrm{d}H\,\mathrm{d}f^{\perp}\Bigg]\mathrm{d}u =$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\Big[\int\limits_{f^\perp\in\mathfrak{p}^\perp}\int\limits_{\{H,X\}\in\mathfrak{p}\oplus\mathfrak{z}} e^{i\cdot\left(f(H)+f^\perp(X)\right)}\phi_u(H,X)\mathrm{d}X\,\mathrm{d}H\,\mathrm{d}f^\perp\Big]\mathrm{d}u,$$

by the Dominated Convergence Theorem, as the integral over $\mathfrak{p}\oplus\mathfrak{z}$ is Schwartz in $f^\perp$. The integral over $\mathfrak{p}^\perp$ amounts to integration over $\mathfrak{z}^*$, by translation invariance in this integral, we may replace $f^\perp(X)$ with $(f+f^\perp)(X)$. As $f(H)=(f+f^\perp)(H)$ for all $f^\perp\in\mathfrak{p}^\perp$, all $H\in\mathfrak{p}$, we may write

$$\mathrm{Tr}(L)=$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\int\limits_{f^\perp\in\mathfrak{p}^\perp}\int\limits_{\{H,X\}\in\mathfrak{p}\oplus\mathfrak{z}} e^{i\cdot(f+f^\perp)(H)}e^{i\cdot(f+f^\perp)(X)}$$
$$\phi\big(\beta(u)\exp(H+X)\beta(u)^{-1}\big)\mathrm{d}X\,\mathrm{d}H\,\mathrm{d}f^\perp\,\mathrm{d}u=$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\int\limits_{f^\perp\in\mathfrak{p}^\perp}\int\limits_{Y\in\mathfrak{g}} e^{i\cdot(f+f^\perp)(Y)}\phi\big(\beta(u)\exp(Y))\beta(u)^{-1}\big)\mathrm{d}Y\,\mathrm{d}f^\perp\,\mathrm{d}u=$$

(letting $Y\to\mathrm{Ad}(\beta(u^{-1}))(Y)$ in the last integral above)

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\int\limits_{f^\perp\in\mathfrak{p}^\perp}\int\limits_{Y\in\mathfrak{g}} e^{i\cdot(f+f^\perp)(\mathrm{Ad}(\beta(u^{-1}))(Y))}\phi(\exp(Y))\mathrm{d}Y\,\mathrm{d}f^\perp\,\mathrm{d}u=$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\int\limits_{f^\perp\in\mathfrak{p}^\perp}\int\limits_{Y\in\mathfrak{g}} e^{i\cdot(\mathrm{Ad}^*(\beta(u))(Y)f+f^\perp)}\phi(\exp(Y))\mathrm{d}Y\,\mathrm{d}f^\perp\,\mathrm{d}u=$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x)\int\limits_{f^\perp\in\mathfrak{p}^\perp}\widehat{\phi}\big(\mathrm{Ad}^*(\beta(u)^{-1})(f+f^\perp)\big)\mathrm{d}f^\perp\,\mathrm{d}u. \tag{11}$$

We now split $\mathfrak{g}/\mathfrak{p}$ into $\mathfrak{g}/\mathfrak{s}$ and $\mathfrak{s}/\mathfrak{p}$, and as $\beta(t)\cdot x=x$ for all $t\in\mathfrak{s}/\mathfrak{p}$, and we get that formula (11) above equals

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x)\int\limits_{v\in\mathfrak{s}/\mathfrak{p}}\int\limits_{f^\perp\in\mathfrak{p}^\perp}\widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\beta(v))(f+f^\perp)\big)\mathrm{d}f^\perp\,\mathrm{d}v\,\mathrm{d}u=$$

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int\limits_{v\in\mathfrak{s}/\mathfrak{p}} \int\limits_{f_2^\perp\in\mathfrak{s}^\perp} \int\limits_{f_1^\perp\in\mathfrak{p}^\perp/\mathfrak{s}^\perp} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\beta(v))(f+f_1^\perp+f_2^\perp)\big)$$
$$\mathrm{d}f_1^\perp\,\mathrm{d}f_2^\perp\,\mathrm{d}v\,\mathrm{d}u. \quad (12)$$

We note that by using the integral in $f_2^\perp$ that we may now assume that f is nonzero only on $\mathfrak{s}$.

We now work with the integral in $f_1^\perp$. If R_f is the stabilizer of the functional f restricted to $\mathfrak{s}$, there exists invariant measures $\mathrm{d}\dot{p}$ and $\mathrm{d}\dot{x}$ on P/R_f and S/P such that $\mathrm{d}\dot{p}\,\mathrm{d}\dot{x}$ is invariant measure on S/R_f. We know from Proposition 109 that $\mathrm{Ad}^*(P)(f) = f+\mathfrak{p}^\perp/\mathfrak{s}^\perp = (f+\mathfrak{p}^\perp)|_\mathfrak{s}$, and that the natural diffeomorphism $\Delta : P/R_f \mapsto \mathrm{Ad}^*(P)(f) = (f+\mathfrak{p}^\perp)|_\mathfrak{s}$ is equivariant and measure preserving on $(f+\mathfrak{p}^\perp)_\mathfrak{s}$. Using different ideas, we may also transfer Euclidean measure $\mathrm{d}f_1^\perp$ on $\mathfrak{p}^\perp/\mathfrak{s}^\perp$ under the translation map q, where $q(f_1^\perp) = f + f_1^\perp$ to a Euclidean measure $\nu = q^*(\mathrm{d}f_1^\perp)$ on the affine space $f+\mathfrak{p}^\perp/\mathfrak{s}^\perp$. Now for each $p \in P$, we define an affine map $A(p)$ from $\mathfrak{p}^\perp/\mathfrak{s}^\perp$ to itself by $A(p)(f_1^\perp) = \mathrm{Ad}^*(p)f_1^\perp + (\mathrm{Ad}^*(p)f - f)$, we note that $(\mathrm{Ad}^*(p)f - f) \in \mathfrak{p}^\perp/\mathfrak{s}^\perp$, see Proposition 109 again. We now note that

$$q\circ A(p)(f_1^\perp) = q\big(\mathrm{Ad}^*(p)(f_1^\perp) + (\mathrm{Ad}^*(p)f - f)\big) = \mathrm{Ad}^*(p)(f+f_1^\perp). \quad (13)$$

As the linear part $\mathrm{Ad}^*(p)|_{\mathfrak{p}^\perp/\mathfrak{s}^\perp}$ of $A(p)$ is unipotent, the operator $A(p)$ preserves $\mathrm{d}f_1^\perp$, and by formula (9) above, $\mathrm{Ad}^*(p)$ preserves ν on the affine space $f+\mathfrak{p}^\perp/\mathfrak{s}^\perp$. As $\mathrm{Ad}^*(p)$ is also measure preserving on P/R_f, we have that under Δ, ν is identified with an invariant measure on P/R_f, which must be a scalar multiple of $\mathrm{d}\dot{p}$: $(\Delta^{-1})^*\nu = c\cdot\mathrm{d}\dot{p}$. Hence if $\phi \in L^1(f+\mathfrak{p}^\perp/\mathfrak{s}^\perp)$ we must have

$$\int\limits_{f_1^\perp\in\mathfrak{p}^\perp/\mathfrak{s}^\perp} \phi(f+f_1^\perp)\mathrm{d}f_1^\perp = \int\limits_{f'\in f+\mathfrak{p}^\perp/\mathfrak{s}^\perp} \phi(f')\mathrm{d}\nu(f') = c\cdot \int\limits_{\dot{p}\in P/R_f} \phi(\mathrm{Ad}^*(\dot{p})f)\mathrm{d}\dot{p}.$$

Consequently, formula (12) above equals

$$\int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{v\in\mathfrak{s}/\mathfrak{p}} \int_{f_2^\perp\in\mathfrak{s}^\perp} c\cdot \int_{m\in\mathfrak{p}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(v))(\mathrm{Ad}^*(\alpha(m))f + f_2^\perp)\big)\, \mathrm{d}m\,\mathrm{d}f_2^\perp\,\mathrm{d}v\,\mathrm{d}u =$$

(by Haar measure we move the $\mathrm{Ad}^*(\alpha(m))$ to include $f_2^\perp$, we note that $\mathrm{Ad}^*(\alpha(m))f_2^\perp$ is still zero on $\mathfrak{s}$)

$$c\cdot \int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{v\in\mathfrak{s}/\mathfrak{p}} \int_{f_2^\perp\in\mathfrak{s}^\perp} \int_{m\in\mathfrak{p}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(v))(\mathrm{Ad}^*(\alpha(m))(f + f_2^\perp))\big)\, \mathrm{d}m\,\mathrm{d}f_2^\perp\,\mathrm{d}v\,\mathrm{d}u =$$

(now combining v and m into a single variable y)

$$c\cdot \int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{f_2^\perp\in\mathfrak{s}^\perp} \int_{y\in\mathfrak{s}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(y))(f + f_2^\perp)\big)\mathrm{d}y\,\mathrm{d}f_2\,\mathrm{d}u.$$

This is our orbital integral. $\square$

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