KIRILLOV THEORY FOR $C^*(G, \Omega)$

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Typeset by $\mathcal{AMS}$-TEX

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Abstract

Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let Ω be a locally compact second countable Hausdorff space with a continuous G action, and let $C^*(G,\Omega)$ be the corresponding transformation group C^* algebra. We construct a continuous surjective map $\phi : \mathfrak{g}^* \mapsto \text{Prim}(C^*(G,\Omega))$ which generalizes the Kirillov correspondance when Ω is a point. We describe an equivalence relation $\sim$ on $\mathfrak{g}^* \times \Omega$ and a map ψ, such that ψ factors through $\mathfrak{g}^* \times \Omega / \sim$ and is a homeomorphism from $\mathfrak{g}^* \times \Omega$ to $\text{Prim}(C^*(G,\Omega))$. We also describe a character theory for $C^*(G,\Omega)$ which generalizes Kirillov character theory for $C^*(G)$.

AMS Subject classifications: 46, 47, 22.

Keywords: Transformation group C^* algebras, Primitive ideal spaces, Nilpotent Lie groups, Kirillov Theory

List of symbols

Prim: Denotes a primitive ideal space

G: Denotes a group, generally nilpotent Lie

$\mathfrak{g}$: Denotes the Lie algebra of G.

$\mathfrak{g}^*$: Denotes the dual space of $\mathfrak{g}$

Ω: Denotes a locally compact second countable Hausdorff space

$C_0(\Omega)$: Denotes the continuous functions vanishing at infinity on Ω

$C_c(\Omega)$: Denotes the continous function with compact support on Ω

$C^*(G,\Omega)$: Denotes the transformation group C^* algebra generated by G and Ω

$\mathcal{K}(G)$: Denotes the set of closed subgroups of G

$\mathfrak{g}$: Denotes a Lie Algebra, generally nilpotent

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ϕ, *psi*: Denote maps

$\sim$, $\sim_1$: Denotes equivalence relations

$\mathcal{K}(G)$: Denotes the set of closed subgroups of G

$\widehat{\ }$: Denotes a primitive ideal or representation space

μ: Denotes the Haar measure on a group

Δ: Denotes the modular function on a group

G_x: Denotes the stabilizer subgroup of $x \in G$

$\mathfrak{g}_x$: Denotes the Lie algebra of G_x

$\prec$: Is used to denote weak containment of representations

$\mathfrak{g}_x$: Denotes the Lie algebra of G_x

$\mathcal{I}(A)$: Denotes the set of closed ideals on a C^* algebra A

$\mathcal{Q}(G)$: Denotes the set of subgroup representation pairs $\{\langle H,T\rangle \mid H \in \mathcal{K}(G),\ T \in \mathrm{Rep}(H)\}$

$\mathcal{Q}(G)$: Denotes the set of subgroup representation pairs

$[x,\ y]$: Denotes the Lie bracket of x and y

$\mathcal{Q}(G)$: Denotes the set of subgroup representation pairs

$\mathfrak{p}_x$: Denotes the polarizing subalgrbra of the subalgebra $\mathfrak{g}_x$ with respect to the action of $f \in \mathfrak{g}_x^*$

ind: Denotes induced representations

$\mathcal{S}(G)$: Denotes the Schwartz functions on a nilpotent Lie group G

$\mathbb{R}$: Denotes the real numbers

$\mathbb{C}$: Denotes the complex numbers

$\mathbb{N}$: Denotes the natural numbers

$\mathcal{N}$: Denotes a locally compact space, for the purpose of this paper, $\mathbb{N} \cup \infty$

$\mathcal{P}$: Denotes any polynomial coordinate map $\psi : \mathbb{R}^n \mapsto G$

$\gamma,\ \alpha,\ \beta, \delta$: Denote maps from real number spaces to G; these are made more clear in the text

$\gamma,\ \alpha,\ \beta, \delta$: Denote maps from real number spaces to G; these are made more clear in the text

$\mathcal{O}_{(f,x)}$: Denotes the orbit of the functional-point pair (f,x) under the action of G; this is defined more clearly in the text

$\mathcal{A}_{G\cdot x}$: Denotes a particular subalgebra of $\mathcal{S}(G) \times C_0(\Omega)$; this is made more clear in the text

Introduction

Our primary references for this paper are the beautiful book of Corwin and Greenleaf [3]; fundamental ideas were provided by Siegfried Echterhoff's paper [7], and the original inspiration was Dana Williams' paper [26].

Let G be a locally compact group acting on a locally compact Hausdorff space Ω. A basic question of C^* theory is an explicit description of $\mathrm{Prim}(C^*(G,\Omega))$. For G a connected, simply connected nilpotent Lie group and Ω a point, a description of $\mathrm{Prim}(C^*(G,\Omega))$ as a set was given in the classic paper of Kirillov [21] where he showed that the natural map from $\mathfrak{g}^*/G \mapsto \widehat{G}$ was continuous with $\mathfrak{g}^*/G$ in the quotient topology. Kirillov conjectured that this was a homeomorphism; this was first proved by Brown [1]. When Ω is not a single point, but G is abelian, a complete description of $\mathrm{Prim}(C^*(G,\Omega))$, including a natural description of the topology, was provided by Dana Williams [26]; in [7], for general G, Siegfried Echterhoff gave a complete description under strong hypotheses for the action.

In this paper we combine the ideas of Williams, Kirillov and Echterhoff to give a natural description of $\mathrm{Prim}(C^*(G,\Omega))$ when G is a connected, simply connected nilpotent Lie group acting on a locally compact Hausdorff space Ω (subject to certain restrictions on the action).

Recall the basic results of Williams [26]. Let G be abelian and $\hat{G}$ be its Pontrjagin dual. Williams produces an equivalence relation $\sim$ on $\hat{G} \times \Omega$ and a well-defined natural map $\psi \;: \hat{G} \times \Omega \mapsto \mathrm{Prim}(C^*(G,\Omega))$ which is a homeomorphism from $\hat{G} \times \Omega/\sim$ to $\mathrm{Prim}(C^*(G,\Omega))$.

Now let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$.

Kirillov theory and the ideas of Dana Williams and Siegfried Echterhoff suggest that there should be a natural map from $\mathfrak{g}^* \times \Omega$ onto $\text{Prim}(C^*(G, \Omega))$. Furthermore, there should be an equivalence relation on $\mathfrak{g}^* \times \Omega$ that subsumes the equivalence relations of Williams in [26], Echterhoff in [7], as well as the coadjoint equivalence relation of Kirillov when Ω is a point.

We define an extended Kirillov map $\phi : \mathfrak{g}^* \times \Omega \mapsto \text{Prim}(C^*(G, \Omega))$, and, with a mild restriction upon the action, show that ϕ is a continuous open surjective map when $\mathfrak{g}^* \times \Omega$ is given the product topology. We further provide a homeomorphism ψ from a quotient space $\sim$ of $\mathfrak{g}^* \times \Omega$ to $\text{Prim}(C^*(G, \Omega))$.

When $C^*(G, \Omega)$ is Type I, we provide a character theory corresponding to the Kirillov character theory of G.

This paper is divided into three sections. Section one consists of preliminary results as well as setting notation. In section two we prove the main results of this paper, *(a)* an explicit parameterization of $\text{Prim}(C^*(G, \Omega))$ via Kirillov theory, and, *(b)* a computation of the topology of $\text{Prim}(C^*(G, \Omega))$. In section three, we generalize the Kirillov character formula for $C^*(G, \Omega)$ under stronger hypotheses of the action.

This paper constitutes part of the author's doctoral thesis at the University of Colorado at Boulder, as well as an abstraction done later. I would like to thank my advisor, Jeff Fox, for his help. I would also like to extend a word of thanks to Larry Baggett, Carla Farsi, Arlan Ramsay, and Marty Walter for helpful discussions. Another word of thanks is extended to Judy Packer, my first doctoral advisor prior to Jeff Fox, for her suggestions and words of encouragement.

Section 1

Preliminaries

Section 1.1

C^* algebras and representation spaces

The purpose of this section is to record some basic facts about representations of transformation group C^* algebras.

Definition 1. Let Ω be a locally compact second countable Hausdorff space. Denote the space of complex valued functions on Ω vanishing at infinity by $C_0(\Omega)$. Denote by $C_c(\Omega)$ the subspace functions in $C_0(\Omega)$ with compact support.

Let G be a locally compact group with a jointly continuous action on Ω, that is, there is a continuous map $G \times \Omega \mapsto \Omega$, where for $r, s \in G$, $r \cdot (s \cdot x) = (rs) \cdot x$, where $s \cdot x$ is the image of (s,x) in Ω.

As (G,Ω) is a locally compact transformation group, we may form the transformation group C^* algebra, denoted $C^*(G,\Omega)$; for a precise treatment of the details, see [6], [8], [18], [25], and [26]. Let μ be a fixed Haar measure on G. Define the Banach $^*-$ algebra $L^1(G,\Omega)$ of all Bochner integrable $C_0(\Omega)$-valued measurable functions on G with respect to μ, with multiplication and involution defined by

$$f * g(s,x) = \int_{r\in G} f(r,x)g(r^{-1}s, r^{-1}\cdot x)\mathrm{d}\mu_G(r)$$

$$f^*(s,x) = \Delta(s^{-1})\overline{f}(s^{-1}, s^{-1}\cdot x),$$

where $f, g \in L^1(G, C_0(\Omega))$, $s \in G$, $x \in \Omega$, Δ is the modular function on G, and $\overline{}$ denotes complex conjugation.

Denote by $C^*(G,\Omega)$ the enveloping C^* algebra of the Banach $^*-$ algebra $L^1(G,\Omega)$ under the multiplication and involution as just defined.

When Ω is a single point, the above construction gives the group C^* algebra of G.

Let $C_c(G,\Omega)$ be the dense subalgebra of $C^*(G,\Omega)$ consisting of continous functions of compact support on $G \times \Omega$.

Denote by Ω/G the orbit space with the quotient topology.

For $\phi \in C_0(\Omega)$, define ${}^s\phi$ by ${}^s\phi(x) = \phi(s^{-1} \cdot x)$.

Denote by $\mathcal{K}(G)$ the space of closed subgroups of G, given the compact Hausdorff topology of Fell [10].

Let $x \in \Omega$; by G_x we denote the stabilizer of x in G. These are assumed connected.

We assume representations non-degenerate.

Definition 2. A *covariant representation* $L = (V, M)$ of (G, Ω) on a Hilbert space H consists of a uniformly bounded strongly continuous unitary representation V of G on H_L, and a norm-decreasing non-degenerate *-preserving representation of $C_0(\Omega)$, M on H_L such that $V(s)M(\phi)V(s^{-1}) = M({}^s\phi)$.

Theorem 1. *A covariant representation $L = (V, M)$ on H_L gives a representation of $L^1(G,\Omega)$, also called L, by*

$$L(f)(\xi) = \int_{s \in G} M(f(s,\cdot))V(s)(\xi)\, d\mu_G(s) \quad (\xi \in H_V).$$

Let A be a fixed C^* algebra. We use the hull-kernel topology on $\widehat{A}$; see [24], Theorem 5.4.6.

On the set $\mathcal{I}(A)$ of closed ideals of A we use the topology developed by Dana Williams on pg. 338 [26]; this is the topology that has as its sub-base the sets $\{\mathcal{O}_J\}_{J\in\mathcal{I}(A)}$, where $\mathcal{O}_J = \{I \in \mathcal{I}(A) \mid I \not\supseteq J\}$. On Prim$(A)$ this topology restricts to the usual hull-kernel (Jacobson) topology. One can see that this topology is almost Fell's "inner hull kernel" topology; see [10].

Definition 3. Let π and ρ be representations of a locally compact group G.

The representation ρ is *weakly contained* in π if every positive definite matrix coefficient $\langle \rho(x)\xi, \xi\rangle$ can be approximated uniformly on compacta of G by finite sums of positive definite matrix coefficients $\langle \pi(x)\xi', \xi'\rangle$. For ρ weakly contained in π, we use the notation $\rho \prec \pi$. The *spectrum* of π is the set of all representations weakly contained in π.

Let $H \subseteq G$ be a closed subgroup, and let f_0 be a nonnegative, real valued function in $C_c(G)$ that does not vanish at the identity element. For the remainder of this paper, let μ_H be the left Haar measure on H defined by

$$\int_H f_0(t)\mathrm{d}\mu_H(t) = 1.$$

Such a choice is referred to as a continuous ("smooth") choice of Haar measures, and has the property that $H \mapsto \int_H f \mathrm{d}\mu_H$ is continuous on $\mathcal{K}(G)$ for each $f \in C_c(G)$; see [15], page 908.

Lemma 1. *Assume $\{f_n\}_{n=1}^{\infty} \subseteq C_c(G)$ converges to $f \in C_c(G)$ in the inductive limit topology and $H_n \to H$ in $\mathcal{K}(G)$. Then*

$$\int_{H_n} f_n \, d\mu_{H_n} \longrightarrow \int_H f \, d\mu_H .$$

Definition 4. Assume that $\{K_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{K}(G)$, with $K_n \to K$ in $\mathcal{K}(G)$. Let $\{\rho_n\}_{n=1}^{\infty}$, each ρ_n defined on K_n, be a sequence of representations. We define a topology in which $\{\rho_n\}_{n=1}^{\infty}$ is allowed to approach a representation ρ defined on K as K_n approaches K in the topology of closed subgroups [10]. For more, see [13]. Define by $\mathcal{Q}(G)$ the set of all subgroup representation pairs of G, where $\langle H, T\rangle$ is such a pair if $H \in \mathcal{K}(G)$ and $T \in \mathrm{Rep}(H)$. We present a topology upon this space.

Lemma 2. *Let $\{\langle K_i, T_i\rangle\}$ be a net of elements of $\mathcal{Q}(G)$ and $\langle K, T\rangle$ an element of $\mathcal{Q}(G)$. Then $\langle K_i, T_i\rangle \to \langle K, T\rangle$ if and only if, for each finite sequence $\phi_1, \cdots, \phi_n$ of functions of positive type on K associated with $\langle K, T\rangle$, and each subnet of $\{\langle K_i, T_i\rangle\}$, there exists (i) a subnet $\{\langle K'^j, T'^j\rangle\}$ of that subnet, and (ii) for each j and each $r = 1, \cdots, n$ a finite sum ϕ_r^j of functions of positive type associated with $\langle K'^j, T'^j\rangle$ such that $\phi_r^i \to \phi_r$ in $\mathcal{E}_s(G)$ for each r.*

Proof.

See [13] Theorem 3.1', page 439. □

Section 1.2

Nilpotent Lie algebras and groups, and their representation spaces

The purpose of this section is to give some basic information about nilpotent Lie algebras and groups that will be needed for this paper.

Let G denote a Lie group and $\mathfrak{g}$ its Lie algebra.

Definition 5. The *adjoint representation*, ad of $\mathfrak{g}$ on $\mathfrak{g}$ is defined as $\mathrm{ad}_x : \mathfrak{g} \mapsto gl(\mathfrak{g})$ by $\mathrm{ad}_x(y) = [x,y]$, for all $y \in \mathfrak{g}$ (here $[\cdot,\cdot]$ denotes Lie bracket on $\mathfrak{g}$).

Definition 6. The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if ad_x is a nilpotent endomorphism of $\mathfrak{g}$, for all $x \in \mathfrak{g}$.

The Lie group G is *nilpotent* if $\mathfrak{g}$ is nilpotent.

Now we briefly describe the representation theory of the nilpotent Lie groups that we use. We follow the notation of Corwin and Greenleaf [3] and use it as our main reference.

Definition 7. Let G be a connected, simply-connected nilpotent Lie group. Denote the dual space of $\mathfrak{g}$ by $\mathfrak{g}^*$.

G acts on $\mathfrak{g}^*$ by the *coadjoint map*, Ad*: for $x \in G$, $Y \in \mathfrak{g}$, and $f \in \mathfrak{g}^*$, define $(\mathrm{Ad}^*(x)f)(Y) = f(\mathrm{Ad}(x^{-1})Y),\ Y \in \mathfrak{g},\ f \in \mathfrak{g}^*,\ x \in G.$

If $f \in \mathfrak{g}^*$. Define the *coadjoint orbit* of f in $\mathfrak{g}^*$ to be $\mathrm{Ad}^*(G)f$.

Let $f \in \mathfrak{g}^*$. A subspace $\mathfrak{p} \subseteq \mathfrak{g}$ is called *isotropic* if $f|_{\mathfrak{p}} = 0$. Let $\mathfrak{p}$ be a maximally isotropic subspace of $\mathfrak{g}$ which is also a subalgebra, then $\mathfrak{p}$ is called a *polarization*, or a *maximal subordinate subalgebra* for f.

For any $M_1, M_2 \in \mathfrak{g}$, we have: $\exp(M_1)\cdot\exp(M_2) = \exp\left(M_1 + M_2 + \frac{1}{2}[M_1,M_2] + \frac{1}{12}[M_1,[M_1,M_2]] + \text{ higher order terms}\right).$

We have:

Lemma 3. *Let $\mathfrak{p}$ be a polarizing subalgebra of $\mathfrak{g}$ for $f \in \mathfrak{g}^*$. Define a one-dimensional representation $\chi_{f,P}$ of $P = exp(\mathfrak{p})$ by $\chi_{f,P}(exp(X)) = e^{i\cdot f(X)}$.*

We may induce the representation $\chi_{f,P}$ from P to a representation $\pi_{f,P}$

of G; for details on induced representations, see [22] and [23].

Give $\mathfrak{g}^*/G$ the quotient topology and $\widehat{G}$ the hull-kernel topology. By [21] and [1] we have the following very important theorem:

Theorem 2.

(1) Let $f \in \mathfrak{g}^*$, and $\mathfrak{p}$ be a polarizing subalgebra for f. Let $\pi_{f,P} = ind_P^G(\chi_{f,P})$. The representation $\pi_{f,P}$ is irreducible, and up to equivalence, every irreducible representation of G is obtained this way.

(1) We have $\pi_{f,P} \cong \pi_{f',P'} \iff \exists\, g \in G$ such that $Ad^*(g)f = f'$.

(3) The induced map (the Kirillov map) $\mathfrak{g}^*/Ad^*(G) \mapsto \hat{G}$ is a homeomorphism.

We often abbreviate $\pi_{f,P}$ as π_f when no confusion arises.

Let G be a nilpotent Lie group, S a closed connected subgroup, having Lie algebras $\mathfrak{g}$ and $\mathfrak{s}$, repectively. Define $\mathfrak{s}^\perp$ to be the set of linear functionals in $\mathfrak{g}^*$ that are zero on $\mathfrak{s}$, i.e., $\mathfrak{s}^\perp = \{f' \in \mathfrak{g}^* \mid f|_\mathfrak{s} = 0\}$.

Lemma 4. *Let G be a simply connected nilpotent Lie group, S a closed connected subgroup; assume $f \in \mathfrak{g}^*$ satisfies $f([\mathfrak{s},\mathfrak{s}]) = 0$ so $\chi_f(exp(Y)) = e^{i\cdot f(Y)}$ is a one dimensional representation of S. The representation $W = ind_S^G(\chi_f)$ weakly contains $\{\pi_{f'} \in \widehat{G} \mid f' \in f + \mathfrak{s}^\perp\}$, in fact, $Sp(W) =$ Fell-closure$\big(\{\pi_{f'} \in \widehat{G} \mid f' \in f + \mathfrak{s}^\perp\}\big)$.*

Proof.

A proof may be found in in [4], Theorem N.2.5.

The following result was proven by Joy [19]. Let H be a subgroup of G.

Let σ be an irreducible representation of H, and $f \in \mathfrak{h}^*$ be associated to σ by the Kirillov map. Denote by Ω_σ the orbit of $f|_{\mathfrak{s}}$ under $\mathrm{Ad}^*(H)$.

Lemma 5. *Let G be a real, connected, simply connected nilpotent Lie group. Let $\langle H_n, T_n\rangle \to \langle H, T\rangle$ in $\mathcal{Q}(G)$. If $f \in \mathfrak{g}^*$ such that $f|_{\mathfrak{h}} \in \Omega_T$, then, for every subsequence of $\{\langle H_n, T_n\rangle\}_{n=1}^{\infty}$, there is a subsequence $\{\langle H_{n_i}, T_{n_i}\rangle\}_{i=1}^{\infty}$ such that for each i, there exists $f_i \in \mathfrak{g}^*$ such that $f_i|_{\mathfrak{h}_{n_i}} \in \Omega_{T_{n_i}}$ and $f_i \to f$ in $\mathfrak{g}^*$.*

Section 1.4

Induced representations and ideals of $C^*(G,\Omega)$

Let G be a group, acting on a locally compact Hausdorff space Ω.

Let $x \in \Omega$ and $G_x = \{g \in G \mid gx = x\}$ be the stability subgroup of x. Let $\rho_x : C_0(\Omega) \mapsto \mathbb{C}$ be the representation of $C_0(\Omega)$ given by evaluation at x; for $\phi \in C_0(\Omega)$, $\rho_x(\phi) = \phi(x)$.

Let τ be a representation of G_x on the Hilbert space H_τ. The pair (τ, ρ_x) forms a covariant pair for $C^*(G_x,\Omega)$ with the representation of $C^*(G_x,\Omega)$ defined as follows: for $v \in H_\tau$, $\phi \in C_0(\Omega)$, $x \in \Omega$, $g \in G_x$, define $\rho_x(\phi)(v) = \phi(x)\cdot v$.

We may induce the representation (τ, ρ_x) from $C^*(G_x,\Omega)$ to $C^*(G,\Omega)$. Let $g \in H_\tau$. Define the induced representation of (τ, ρ_x) to be $L = (V, M)$, where $V = \mathrm{ind}_{G_x}^{G}(\tau)$, and M is a representation of $C_0(\Omega)$ on H_V, acting by $M(\phi)(g)(r) = \psi(r\cdot x)f(r)$. We use the notation $L = (V, M) = \mathrm{ind}_{(S,\Omega)}^{(G,\Omega)}(\tau, \rho_x)$.

We now present a different, equivalent way, to induce representations of $C^*(G,\Omega)$.

Assume H is a closed subgroup of G.

Assume that we have the C^* algebra $C^*(G,\Omega)$, and that π is a representation of $C^*(H,\Omega)$ acting on the Hilbert space H_π. Let ξ and η be arbitrary vectors in H_π. Define the induced representation, $L = \mathrm{ind}_{(H,\Omega)}^{(G,\Omega)}(\pi)$ as follows:

Let $B = C_c(H,\Omega)$, and define a $C_c(H,\Omega)$-valued inner product on $C_c(G,\Omega)$ as follows:

$$\langle f,g\rangle_B(t,y) = \int\limits_{s\in G} \overline{f}(s,s\cdot y)g(st,s\cdot y)\mathrm{d}\mu_G(s). \tag{1}$$

Define an inner product on $C_c(G,\Omega)\otimes H_\pi$ by

$$\langle f\otimes\xi, g\otimes\eta\rangle_L = \langle \pi(\langle g,f\rangle_B)\xi,\eta\rangle_\tau. \tag{2}$$

Let H_L be the completion of $C_c(G,\Omega)\otimes H_\pi$.

The induced representation L of $h\in C^*(G,\Omega)$ acts on the class of $f\otimes\xi$ by $(h*f)\otimes\xi$; see [18], page 204, or [26], page 340.

Proposition 1. *Let G be a group acting on a locally compact Hausdorff space Ω. Fix a point $x\in\Omega$. Let τ a representation of G_x, and ρ_x be a point evaluation of Ω. If τ is irreducible, then the induced representation $L = ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau,\rho_x)$, of $C^*(G,\Omega)$, is an irreducible representation of $C^*(G,\Omega)$.*

Proof.

See [26], Proposition 4.2 on pg. 344. □

Definition 8. Let $L = (V, M)$ be a representation of (H, Ω), $s \in G$. Let $K = sHs^{-1}$, and let $L^s = (V, M)^s = (V^s, M^s)$ be the covariant representation of (K, Ω) given by $V^s(r) = V(s^{-1}rs)$, and $M^s(\phi) = M({}^s\phi)$.

The following has second-countability as a hypothesis, true in our case.

Theorem 3. *If* $S = ind_{(H,\Omega)}^{(G,\Omega)}(V, M)$ *and* $T = ind_{(K,\Omega)}^{(G,\Omega)}((W, N))$*, then* $S \cong T$ *if and only if for some* $s \in G$*, we have* $T = ind_{(sKs^{-1},\Omega)}^{(G,\Omega)}((V, M)^s)$*.*

Proof.

See [15], Theorem 2.1. □

We shortly prove an important proposition that enables us to separate certain kernels of $C^*(G, \Omega)$. Several definitions are needed; these definitions will be used for the remainder of this paper.

Definition 9. For $x \in \Omega$, $f \in \mathfrak{g}^*$, G_x will denote the stabilizer of x in G; $\mathfrak{p}_x$ will denote a polarizing subalgebra of $\mathfrak{g}_x$ for f. Denote by $\mathfrak{p}$ an isotropic subalgebra, not necessarily polarizing. Also define

χ_{f,P_x} to be the character of $P_x = \exp(\mathfrak{p}_x)$ given by $\chi_{f,P_x}(\exp(X)) = e^{i \cdot f(X)}$

$\tau_{f,x} = \mathrm{ind}_{P_x}^{G_x}(\chi_{f,P_x})$, an irreducible representation of G_x.

The following equivalence relation $\sim_1$ was motivated by D. Williams [26], and reduces to his when G is abelian.

Definition 10. Define an equivalence relation $\sim_1$ on $\mathfrak{g}^* \times \Omega$ as follows: $(f, x) \sim_1 (f', x')$ when:

(1) There exists $s \in G$ such that $x' = s \cdot x$.

(2) For some $h \in \mathfrak{g}_{x'}^{\perp}$, we have $\mathrm{Ad}^*(s)f = f' + h$.

We write $\mathcal{O}_{(f,x)}$ for the equivalence class of (f,x).

Now we define a second quotient space $\sim$ on $\mathfrak{g}^* \times \Omega$ by $\mathcal{O}_{(f,x)} \sim \mathcal{O}_{(h,y)} \iff \overline{\mathcal{O}_{(f,x)}} = \overline{\mathcal{O}_{(h,y)}}$. This is denoted $\mathfrak{g}^* \times \Omega/\sim$.

Let (V, M) be a covariant pair for (G, Ω). If M is given by a point evaluation ρ_x of a point $x \in \Omega$, i.e., if for $\phi \in C_0(\Omega)$ we have $M(\phi)(r) = \phi(r \cdot x)$, we often write (V, ρ_x) for (V, M).

Proposition 2 which follows does not depend upon G being a nilpotent Lie group. It has been proven by Takesaki ([25], Theorem 7.2) when our orbit space Ω/G has a T_0 topology. It was proven in generality by Phil Green [18], pg. 210 Proposition 11, Part (ii). We present a different proof for our case.

As Ω is assumed separable, it is metrizable by [20], page 146 and Theorem 16, page 125. Hence Ω is T_4, and the Tietze Extension Theorem be applied to Ω; this is needed to prove the next lemma.

Lemma 6. *Let $x \in \Omega$, G_x = the stabilizer of x in G. Let $C \subseteq G/G_x$ be compact in G/G_x. If $f \in C_c(G/G_x)$ is supported on C, we may find a sequence of continuous functions in $C_C(\Omega)$ which converge pointwise to*

$$f_n(y) \to h = \begin{cases} f(s) \text{ when } y = s \cdot x, & y \in C \cdot x \\ 0 & y \notin C \cdot x. \end{cases}$$

Proposition 2. *Let $C^*(G, \Omega)$ be any locally compact transformation group C^* algebra. Let $x \in \Omega$, and let τ_1 and τ_2 be two representations of G_x such* *that (τ_1, ρ_x) and (τ_2, ρ_x) do not have the same kernel as representations of $C^*(G_x, \Omega)$. Defining $L_1 = ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau_1, \rho_x)$ and $L_2 = ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau_2, \rho_x)$, we have $ker(L_1) \neq ker(L_2)$.*

Proof.

Define a new C^* algebra, $C^*(G, G/G_x)$. Our orbit space $(G/G_x)/G$ for $C^*(G, G/G_x)$ is a T_0 space; it consists of one point. Hence by Theorem 3.3 [16], $C^*(G, G/G_x)$ is Type I, and there is canonical homeomorphism between $C^*(\widehat{G, G/}G_x)$ and $\mathrm{Prim}(C^*(G, G/G_x))$.

Identify $x \in \Omega$ with $x' = e \in G/G_x$ in the new orbit space. Observe that $G_{x'} = G_x$. Use the same representations τ_1 and τ_2 of $G_{x'}$. Define $L_1' = \mathrm{ind}_{(G_{x'},G/G_x)}^{(G,G/G_x)},(\tau_1, M_{x'})$ and $L_2' = \mathrm{ind}_{(G_{x'},G/G_x)}^{(G,G/G_x)}(\tau_2, M_{x'})$, two irreducible representations of $C^*(G, G/G_x)$; these have different kernels by Takesaki [25] Theorem 7.2.

Assume for $i = 1, 2$ that ξ_i is a vector in the space of τ_i. The representation $(\tau_i, M_{x'})$ of $C^*(G_{x'}, G/G_{x'})$ we refer to as π_i.

As $\ker(L_1') \neq \ker(L_2')$, find an $h' \in C^*(G, G/G_{x'})$ with $h' \in \ker(L_1')$ and $h' \notin \ker(L_2')$.

Assume that for some elementary tensor $f' \otimes \xi_2$ that $L_2(h')(f' \otimes \xi_2) \neq 0$. Further assume that f' is in the dense subset of continuous functions with compact support having the form $f'(s, y) = \sum_{i=1}^{l} f_{1,i}'(s) f_{2,i}'(y)$.

Assume that f' and its component functions are collectively supported on $C_1 \times C_2$, where C_1 is compact in G and C_2 is compact in G/G_x.

Refer to all inner products of $C^*(G, G/G_x)$ as $\langle \cdot\ \cdot \rangle_i'$.

Assume by scaling arguments that $\langle h' * f' \otimes \xi_1, f' \otimes \xi_1 \rangle_1' = 0$, and $\langle h' * f' \otimes \xi_2, f' \otimes \xi_2 \rangle_2' = 1$.

Assume $\{h_n'\}_{n=1}^{\infty} \subseteq C_c(G, G/G_x)$, and $h_n' \to h'$ in the topology of $C^*(G, G/G_x)$. Assume each h_n' has the form $h_n'(s,y) = \sum_{i=1}^{N_n} \phi_{n,i}'(s)\psi_{n,i}'(y)$, and each is supported on $C_1^n \times C_2^n$, each C_1^n compact in G and each C_2^n compact in G/G_x.

For each n, $h_n' * f'$ is a continuous function of compact support; see [8], pp. 32-33 for a proof.

We have:

$$\langle h_n' * f' \otimes \xi_1, f' \otimes \xi_1 \rangle_1' \to 0$$

$$\langle h_n' * f' \otimes \xi_2, f' \otimes \xi_2 \rangle_2' \to 1.$$

Now return to $C^*(G,\Omega)$. All inner products in this C^* algebra we refer to by $\langle \cdot\ \cdot \rangle_i$.

Let

$$L_1 = \mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_1, \rho_x),$$

and

$$L_2 = \mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_2, \rho_x).$$

Use Lemma 6 to define a collection of functions $\left\{\{h_{n,j}\}_{n,j=1}^{\infty}, \{f_j\}_{j=1}^{\infty}\right\}$ in $C_c(G,\Omega)$. Define $f_j(s,y)$ by

$$f_j(s,y) = \sum_{i=1}^{l} f_{1,i}(s) f_{2,i,j}(y),$$

each $f_{1,i} = f_{1,i}'$; we do not change these functions. Choose $f_{2,i,j}$ to satisfy $f_{2,i,j}(s \cdot x) = f_{2,i}'(s \cdot x')$, so $f_{2,i,j}$ "behaves the same as $f_{2,i}'$" on the set $C_2 \cdot x$, extend to all Ω, and require

$$f_{2,i,j}(y) \underset{j\to\infty}{\longrightarrow} \begin{cases} f'_{2,i}(s\cdot x') \text{ when } y = s\cdot x, \quad s\in C_2 \\ \quad 0 \qquad\quad \text{otherwise,} \end{cases}$$

Note:

$$\lim_{j\to\infty} f_j(s,y) = \begin{cases} f'(s, s\cdot x') \text{ when } y = s\cdot x, \quad s\in C_2 \\ \quad 0 \qquad\quad \text{otherwise,} \end{cases}$$

Similarly define sequences $\{h_{n,j}\}_{n,j=1}^{\infty}$ with (for each fixed n)

$$h_{n,j}(s,y) \underset{j\to\infty}{\longrightarrow} \begin{cases} h'_n(s, s\cdot x') \text{ when } y = s\cdot x, \quad s\in C_2 \\ \quad 0 \qquad\quad \text{otherwise.} \end{cases}$$

For each n and all j (fixed n), assume that the functions f_j and $h_{n,j}$ have support contained in the set $S_1^n \times S_2^n$, S_1^n compact in G, and S_2^n compact in Ω. For each fixed n, choose them uniformly bounded for all j.

Re-write our old calculations with these new functions, and follow the same steps in the C^* algebra $C^*(G,\Omega)$ with these new functions.

"Untwisting" the integral (see formulas (1) and (2)), note that it is in s over a compact set.

Apply the Bounded Convergence Theorem to the integral. These functions are converging in j, and for each n, have been choosen in a bounded fashion with compact support, hence are bounded above by an integrable function. Mixing inner products in the next formula, for each fixed n, limiting $j\to\infty$, we have:

$$\langle h_{n,j} * f_j \otimes \xi_i, f_j \otimes \xi_i\rangle_i \longrightarrow \begin{cases} \langle h'_n * f' \otimes \xi_1, f' \otimes \xi_1\rangle'_1 \quad (i=1) \\ \langle h'_n * f' \otimes \xi_2, f' \otimes \xi_2\rangle'_2 \quad (i=2). \end{cases}$$

We remind the reader that these inner products are in the C^* algebra $C^*(G, G/G_x)$.

Assume (fixed n) that for all $j \geq n$

$$\left|\langle h_{n,j} * f_j \otimes \xi_1, f_j \otimes \xi_1\rangle_1 - \langle h'_n * f' \otimes \xi_1, f' \otimes \xi_1\rangle'_1\right| < \frac{1}{n}$$
$$\left|\langle h_{n,j} * f_j \otimes \xi_2, f_j \otimes \xi_2\rangle_2 - \langle h'_n * f' \otimes \xi_2, f' \otimes \xi_2\rangle'_2\right| < \frac{1}{n},$$

Choose the diagonal sequence $\{h_{j,j}\}_{j=1}^{\infty}$ and the sequence $\{f_j\}_{j=1}^{\infty}$ in the above a sequences, and for $i = 1$ the sequence converges to 0, for $i = 2$ it converges to 1. As ξ_1 was arbitrary, we have $L_1(h_{j,j}) \to 0$ and $L_2(h_{j,j}) \not\to 0$, showing that L_1 and L_2 cannot have the same kernel. □

Corollary 1. Assume the pairs (f_1, x) and (f_2, x) give rise to irreducible representations (τ_1, ρ_x) and (τ_2, ρ_x) of $C^*(G_x, \Omega)$. These induce to equivalent irreducible representations of $C^*(G, \Omega)$ if and only if τ_1 and τ_2 are the equivalent irreducible representations of G_x. This says that (f_1, x) and (f_2, x) are in the same equivalence class mod $\sim_1$, and $f_1|_{\mathfrak{g}_x}$ and $f_2|_{\mathfrak{g}_x}$ are in the same G_x orbit in $\mathfrak{g}_x^*$; see Definition 10.

For the remainder of this section we use Siegfried Echterhoff [7] as our main reference.

Definition 11. Let G be our connected, simply-connected nilpotent Lie group, and let $\mathcal{K}(G)$ be the space of closed subgroups of G. Let $\mathcal{N}$ be a locally compact space. Assume $H : \mathcal{N} \mapsto \mathcal{K}(G)$ and $H(i) = H_i$ is a continuous map from $\mathcal{N}$ to $\mathcal{K}(G)$. Define:

$$\mathcal{N}^H = \{(i, x) \in \mathcal{N} \times G \mid x \in H_i\}$$

In this paper, we use $\mathcal{N} = \mathbb{N} \cup \infty$, the one-point compactification of the natural numbers.

The following results are from [7].

Definition 12.

Now let G again be a nilpotent Lie group and (G,Ω) denote a covariant system. We make the space $C_c(\mathcal{N}^H, C_0(\Omega))$ into a normed *- algebra. Define multiplication, involution, and norms by

$$\begin{aligned} f * g(i,t,x) &= \int\limits_{s\in H_i} f(i,s,x)g(i,s^{-1}t,s^{-1}\cdot x)\mathrm{d}\mu_{H_i}(s) \\ f^*(i,t,x) &= \overline{f}(i,t^{-1},t^{-1}\cdot x) \\ \|f\|_1 &= \sup_{i\in\mathcal{N}} \int\limits_{s\in H_i} \sup_{x\in\Omega} |f(i,s,x)|\mathrm{d}\mu_{H_i}(s) \end{aligned}$$

for $f, g \in C_c(\mathcal{N}^H, C_0(\Omega))$.

Definition 13. Denote by $L^1(\mathcal{N}^H, C_0(\Omega))$ the completion of $C_c(\mathcal{N}^H, C_0(\Omega))$ with respect to the above norm. Any covariant representation τ of the system $\big(H_i, C_0(K_i, C_0(\Omega))\big)$ defines a * - representation of $L^1(\mathcal{N}^H, C_0(\Omega))$ by

$$(i,\tau)(F) = \tau(F_i)$$

Definition 14. The representation space

$$\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))) = \big\{(i,\rho) \mid i \in \mathcal{N}, \rho \in \mathrm{Rep}(H_i, C_0(\Omega))\big\}$$

with the relative topology of $\text{Rep}(C^*(\mathcal{N}^K, C_0(\Omega)))$ is the *subgroup representation space* of $C^*(\mathcal{N}^K, C_0(\Omega))$.

Let $C^*(\mathcal{N}^H, C_0(\Omega))$ be the enveloping C^* algebra of $L^1(\mathcal{N}^H, C_0(\Omega))$ We call $C^*(\mathcal{N}^H, C_0(\Omega))$ the *subgroup algebra* of $(\mathcal{N}^H, C_0(\Omega))$.

A standard technique in nilpotent harmonic analysis is induction from codimension one subgroups. The next proposition extends this idea from the above-defined algebras to our transformation group C^* algebras.

Proposition 3. *Let G be a nilpotent Lie group acting upon the locally compact Hausdorff space Ω.*

(1) Assume we have two continuous maps, $k(n)$ and $h(n)$, from $\mathcal{N} = \mathbb{N} \cup \infty$ to $\mathcal{K}(G)$. Also assume that for all n, $h(n) = H_n$ is codimension 1 in $k(n) = K_n$.

(2) Assume we have a collection of representations $\{\{(i, \pi_i)\} \mid i \in \mathcal{N}\}$ in the space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$, and, for each $i \in \mathbb{N}$, π_i is induced from (H_i, Ω). Assume for each $i \in \mathbb{N}$ that π_i is irreducible on $C^*(K_i, \Omega)$, and that each π_i corresponds to a functional-point pair (f_i, x_i).

(3) Assume in the space $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega)))$ that $(i, \pi_i) \to (\infty, \pi_\infty)$.

Then,

(1) By passing to a subsequence, we may choose a collection $\{\sigma_i\}_{i=1}^{\infty}$, with each σ_i being an irreducible representation of $C^*(H_i, \Omega)$, and σ_∞, an irreducible representation of (H_∞, Ω), these representations satisfying:

$$ker(\pi_i) = ker(ind_{(H_i,\Omega)}^{(K_i,\Omega)}(\sigma_i)),$$

$$\pi_\infty \prec \mathit{Ind}_{(H_\infty,\Omega)}^{(K_\infty,\Omega)}(\sigma_\infty), \text{ and } (i,\sigma_i) \to (\infty,\sigma_\infty) \text{ in } \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega))).$$

(2) For each i, the functional-point pairs (Definition 10) corresponding to σ_i may be chosen in the same equivalence class mod $\sim_1$ as (f_i, x_i).

Proof.

By assumption, for each $i \in \mathbb{N}$, π_i is induced from (H_i,Ω), so assume that $\pi_i = \mathrm{ind}_{(H_i,\Omega)}^{(K_i,\Omega)}(\sigma_i)$. By Corollary 1, we may assume that the functional-point pairs (Definition 10) corresponding to σ_i are the same as those corresponding to π_i.

The restriction map $\mathcal{R}(C^*(\mathcal{N}^K, C_0(\Omega))) \mapsto \mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ is continuous (Proposition 7 of [7]), so we have:

$$(i, \pi|_{(H_i,\Omega)}) \underset{i\to\infty}{\longrightarrow} (\infty, \pi|_{(H_\infty,\Omega)}).$$

Note $H(i)$ is normal in $K(i)$ ([3], Lemma 1.1.8). By [26], Proposition 7 on pg. 70, for each i, the restricted representation is equivalent to the following direct integral:

$$(i, \pi_i|_{(H_i,\Omega)}) = (i, \int_{s\in K_i/H_i}^{\oplus} \sigma_i^s \ ).$$

Prior to finishing the proof we present several facts particular to connected, simply connected nilpotent Lie groups. We observe that our group representations are not generally irreducible, but are induced from irreducible representations of subgroups, hence are direct integrals of irreducibles, as are their restrictions to subgroups.

Fact 1: Let $\pi = \pi_f$ be an infinite-dimensional irreducible representation of G. Let G_0 be a codimension one subgroup of G, and let $f_0 = f|_{\mathfrak{g}_0}$ the restriction of f to $\mathfrak{g}_0$. Let $\pi_0 = \pi_{f_0}$ be the irreducible representation of G_0 associated by Kirillov theory to f_0 on $\mathfrak{g}_0$.

By [3], Theorem 2.5.3, the representation $\pi|_{G_0}$ is either a representation that induces directly to π, or is the following direct integral:

$$\pi|G_0 = \int_{s\in G/H}^{\oplus} (\pi_0)^s,$$

and $\pi = \mathrm{ind}_{G_0}^G(\pi_0)$.

Let $\sigma = \mathrm{ind}_{G_0}^G(\pi_0)$. We have one of the following:

a) $\sigma \cong \pi$

b) $\sigma \cong \infty \cdot \pi$.

Fact 2: Recall Lemma 2.4 on pg. 338 of [26]:

Lemma. *Let A be a C^* algebra. Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a net of ideals of A, converging to I. Assume that each I_α is the intersection of a set of primitive ideals $F_\alpha \in \mathcal{K}(Prim(A))$, and F corresponds to I. Then, given any $P \supseteq F$, there is a subnet of primitive ideals of A, $\{I_\beta\}_{\beta\in\Lambda'}$, such that there are $P_\beta \in F_\beta$, with $\{P_\beta\}_{\beta\in\Lambda'}$ converging to P in $Prim(A)$.*

Now we return to the proof of our proposition.

Note that π_∞ may be one-dimensional, hence not induced from H_∞.

By the first of the above two facts, choose an irreducible representation σ'_∞ of $C^*(H_\infty,\Omega)$ with $\sigma'_\infty \prec \pi_\infty|_{(H_\infty,\Omega)}$ and $\pi_\infty \prec \mathrm{ind}_{(H_n,\Omega)}^{(K_n,\Omega)}(\sigma'_\infty)$.

We may assume that the functional-point pair for σ'_∞ is in the same $\sim_1$ class as that of π_∞.

By an application of the second fact above, we may pass to a subsequence, and choose $\{(n,\sigma'_n)\}_{n=1}^\infty$ with $(n,\sigma'_n) \to (n,\sigma'_\infty)$.

By an application of Theorem 3 and Corollary 1, for all $n \in \mathbb{N}$, we have $\sigma'_n \cong \sigma_n^{s_n}$ for some $s_n \in H_n$. So the functional-point pairs associated to σ_n and σ'_n are in the same equivalence class mod $\sim_1$. □

Lemma 7. *Assume that G is a nilpotent Lie group acting on a locally compact Hausdorff space Ω.*

Assume we have a continuous map $h(i) = H_i$ from $\mathcal{N} = \mathbb{N} \cup \infty$ to $\mathcal{K}(G)$ with $h(i) = H_i$, and $H_i \to H_\infty$.

Assume we have a collection of one-dimensional representations $\big\{\{\pi_i \mid i \in \mathcal{N}\}\big\}$, $\pi_i \in Rep(C^*(H_i,\Omega))$, each $\pi_i = (\chi_{f_i}, M_{x_i})$.

Assume in the space $\mathcal{R}(C^*(\mathcal{N}^H, C_0(\Omega)))$ that $(i,\pi_i) \to (\infty,\pi_\infty)$.

Then:

(a) $x_n \to x$, *and*

(b) *by passing to a subsequence, we may choose another collection $\{f'_i\}_{i=1}^\infty$ with $\chi_{f'_i} = \chi_f$.*

Furthermore, for each $i \in \mathcal{N}$, (f'_i, x_i) may be chosen in the same $\sim_1$ class of $\mathfrak{g}^* \times \Omega$ as (f_i, x_i).

Proof.

For any $\phi \in C_c(G)$ and $\psi \in C_c(\Omega)$, we may consider $\phi{\cdot}\psi \in C^*(\mathcal{N}^H, C_0(\Omega))$ by setting $(\phi \cdot \psi)(i) = \phi|_{H_i} \cdot \psi$. By hypothesis,

$$\pi_i(\phi|_{H_i} \cdot \psi) = \psi(x_i) \int\limits_{s \in H_i} \phi(s)\chi_{f_i}(s)\mathrm{d}\mu_{H_i}(s)$$
$$\longrightarrow \psi(x) \int\limits_{s \in H} \phi(s)\chi_f(s)\mathrm{d}\mu_H(s) = \pi_\infty(\phi|_{H_\infty} \cdot \psi).$$

Let e denote the identity element of the group. By the continuity of $\mathrm{Res}_e^{H_n}$, the the restriction map from $\mathcal{R}(C^*(\mathcal{N}, C_0(\Omega)))$ to Ω, (Proposition 7 of [7]), we have for $\phi \in C_0(\Omega)$, $\mathrm{Res}_e^{H_n}(\pi_i)(\phi) = \phi(x_i)$, and $\phi(x_i) \to \phi(x)$ for all $\phi \in C_0(\Omega)$; consequently $x_i \to x$.

Choose ψ identical one in a neighborhood of $x \in \Omega$. In Fell's subgroup-pair topology [12] we have:

$$\langle \chi_{f_i}, H_i \rangle \to \langle \chi_f, H_\infty \rangle.$$

Pass to a subsequence, and use Lemma 5 to find a sequence $\{f_i'\}_{i=1}^\infty \subseteq \mathfrak{g}^*$ with $\chi_{(f_i'|_{H_i})} = \chi_{(f_i|_{H_i})}$ and $f_i' \to f_\infty$ in $\mathfrak{g}^*$ and by Lemma 5, each f_i' is in the same H_i-orbit as f_i and (f_i', x_i) is in the same $\sim_1$ equivalence class of $\mathfrak{g}^* \times \Omega$. □

Now for our "big" lemma, which is very important in the next section.

Lemma 8. *Assume that we have a collection* $\{\{P_i,\}_{i=1}^\infty, P\}$ *of primitive ideals of* $C^*(G,\Omega)$*, with*

$$P_i = ker(L_i) = ker(ind_{(G_{x_i},\Omega)}^{(G,\Omega)}(\tau_{f_i,x_i}, M_{x_i}))$$
$$\longrightarrow P = ker(L) = ker(ind_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, \rho_x)).$$

By passing to a subsequence we may choose:

$$\{y_i\}_{i=1}^{\infty} \subseteq \Omega \text{ and } \{f_i'\}_{i=1}^{\infty} \subseteq \mathfrak{g}^*, \text{ with } y_i \to y \in \Omega, f_i' \to f',$$

with

$$ker(L_i) = ker(ind_{(G_{y_i},\Omega)}^{(G,\Omega)}(\tau_{f_{ii}',y_i}, M_{y_i}))$$

and

$$ker(L) = ker(ind_{(G_{x'},\Omega)}^{(G,\Omega)}(\tau_{f'}, M_y)).$$

Furthermore, the functional-point pairs (f_i', y_i) corresponding to τ_i' may be chosen in the same $\sim_1$ equivalence class as (f_i, x_i), and (f', y) in the same $\sim_1$ equivalence class as (f, x).

Proof.

By Lemma 4.5 of [26] we may assume that $x_i \to x$ and that the $\sim_1$ equivalence classed don't change.

If an infinite subsequence of $\{L_i\}_{i=1}^{\infty}$ consists of one-dimensional representations, we may use Lemma 7.

If no infinite subsequence of $\{L_i\}_{i=1}^{\infty}$ consists of one-dimensional representations, we may assume that each is induced from a codimension one subgroup H_i. We pass to a subsequence, and assume that for the sequence $\{\mathfrak{p}_i\}_{i=1}^{\infty}$ of polarizing subalgebras, we have $\dim(\mathfrak{p}_i)$ constant.

By compactness of $\mathcal{K}(G)$, we may pass to a subsequence and assume that $G_{x_i} \to S \subseteq G_x$ and $\mathfrak{p}_i \to \mathfrak{p}$, where $\mathfrak{p}$ may not be polarizing for the action of f on $\mathfrak{g}_x$. Assume that a representation at least weakly containing L is induced from (H,Ω).

We may use Proposition 3 to succesively reduce our problem by one dimension, until we do have a sequence of one-dimensional representations, and Lemma 7 may be applied to these. $\square$

Section 3

The Topology on $\mathbf{Prim}(C^*(G,\Omega))$

Lemma 9. *Let $x \in \Omega$, $f \in \mathfrak{g}^*$ be given.*

Let $\mathfrak{p}_x$ be polarizing for the restriction of f to $\mathfrak{g}_x$, and let $\mathfrak{p}$ be isotropic (not necessarily polarizing) for the restriction of f in $\mathfrak{g}_x$. Then $L' = (V', M) = ind_{(P,\Omega)}^{(G,\Omega)}(\chi_{f,P}, x)$ weakly contains $L = (V, M) = ind_{(P_x,\Omega)}^{(G,\Omega)}(\chi_{f,P_x}, x)$.

Proof.

Let $\chi_{f,P}$ be the character of P determined by f, and let χ_{f,P_x} be the corresponding character of P_x. By Lemma 4, on the level of stabilizer subgroups, we have $\mathrm{ind}_{(P_x,\Omega)}^{(G_x,\Omega)}(\chi_{f,P_x}, \rho_x) \prec \mathrm{ind}_{(P,\Omega)}^{(G_x,\Omega)}(\chi_{f,P}, \rho_x)$. As induction preserves weak containment (Proposition 9, [18]), the conclusion is clear from this and "induction in stages"; see Proposition 8, pg. 207 of [18]. $\square$

Definition 15. Define $\phi : \mathfrak{g}^* \times \Omega \mapsto \mathrm{Prim}(C^*(G,\Omega))$ by

$$\phi(f,x) = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, \rho_x)).$$

Lemma 10. *ϕ is continuous in the product topology of $\mathfrak{g}^* \times \Omega$.*

Proof.

Assume in the product topology of $\mathfrak{g}^* \times \Omega$ that $(f_n, x_n) \to (f, x)$. Denote the sequence of stability subgroups as $\{G_{x_n}\}_{n=1}^{\infty}$; we assume that $G_{x_n} \to S \subseteq G_x$.

Let $\{\mathfrak{p}_n\}_{n=1}^{\infty}$ denote the sequence of polarizing subalgebras of $\mathfrak{g}_{x_n}$ for the sequence of functionals $\{f_n\}_{n=1}^{\infty}$. As $f_n \to f$, we may assume that $\mathfrak{p}_n \to \mathfrak{p}$, where $\mathfrak{p}$ may be of lower dimension than a polarizing subalgebra $\mathfrak{p}_x$ of $\mathfrak{g}_x$ for $f|_{g_x}$.

Denote by g the restricted functional $f|_{\mathfrak{p}}$. Denote by σ the representation of $C^*(P,\Omega)$ given by the obvious character χ_g of P and a point evaluation ρ_x of Ω.

For each n denote by π_n the representation of $C^*(P_{x_n},\Omega)$ given by the pair (χ_{f_n}, M_{x_n}), and by π the representation of $C^*(P_x,\Omega)$ given by the pair (χ_f, ρ_x).

Define $L_n = \mathrm{ind}_{(G_{x_n},\Omega)}^{(G,\Omega)}(\pi_n)$, an irreducible representation of $C^*(G,\Omega)$, and $L = \mathrm{ind}_{(P_x,\Omega)}^{(G,\Omega)}(\rho)$, also an irreducible representation of $C^*(G,\Omega)$.

Let $L' = \mathrm{ind}_{(P,\Omega)}^{(G,\Omega)}(\sigma)$ be induced from $C^*(P,\Omega)$; this may not be irreducible.

Amending Echterhoff [7], Proposition 6 on pg. 69 slightly for our purposes, we have:

Let (G,Ω) be a covariant system, $\mathcal{N}$ be the locally compact space $\mathbb{N}\cup\infty$, and $P(n) = \mathfrak{p}_{x_n} = P_n$ be a continuous map with $P(n) = \mathfrak{p}_n \to \mathfrak{p} = P(\infty)$ in $\mathcal{K}(G)$. Then the map

$$Ind_P^G : \mathcal{R}(C^*(\mathcal{N}^p,\Omega)) \mapsto \widehat{C^*(G,\Omega)}; \quad (n,\pi_n) \mapsto (n, ind_{(P_n,\Omega)}^{(G,\Omega)}(\pi_n))$$

is continuous.

Using this, we have $L_n \to L'$, but as $L \prec L'$ (Lemma 9), we are done. □

The same techniques will yield a simpler proof of [26], Lemma 4.9.

Definition 16. We say that $C^*(G,\Omega)$ is *EH regular* if:

(1) $C^*(G,\Omega)$ is quasi-regular,

(2) for every $P \in \mathrm{Prim}(C^*(G,\Omega))$, there is an $x \in \Omega$ and an irreducible representation τ of G_x such that $P = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau,\rho_x))$.

This has been established in our case by Jon Rosenberg and Elliot Gootman [17].

Remember our equivalence relation $\mathfrak{g}^* \times \Omega/\sim$, as well as the equivalence class $\mathcal{O}_{(f,x)}$ of (f,x) in $\mathfrak{g}^* \times \Omega$; see Definition 10.

Definition 17. Define $\psi \ : \mathfrak{g}^* \times \Omega/\sim\ \mapsto \mathrm{Prim}(C^*(G,\Omega))$ by $\psi(f,x) = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x},\rho_x)$. We show that ψ factors through $\sim$ in the next lemma.

Lemma 11. *The map ψ is one-to-one onto on $\mathfrak{g}^* \times \Omega/\sim$.*

Proof.

Onto is clear by EH regularity; we show that ψ is 1-1.

When $\overline{\mathcal{O}_{(f,x)}} = \overline{\mathcal{O}_{(h,y)}}$, that the primitive ideals defined by (f,x) and (h,y) are the same follows from the continuity of the map ϕ (Lemma 10) and the fact that primitive ideal spaces are T_0.

Now assume that $\psi(f,x) = \psi(h,y)$.

As $\overline{G\cdot x} = \overline{G\cdot y}$, by Lemma 4.5 [26], we may find a sequence $\{g_n\}_{n=1}^{\infty} \subseteq G$ such that $g_n \cdot x \to y$.

Define $g_n \cdot (f,x) = (\mathrm{Ad}^*(g_n)f, g_n \cdot x)$.

We have: $\psi(g_n \cdot (f,x)) = \psi(f,x)$, and the sequence $\psi(g_n \cdot (f,x))$ always remains in the $\sim_1$ equivalence class of (f,x), and $\psi(g_n \cdot (f,x)) = \psi(h,y)$.

If a subsequence of the sequence $\psi(g_n \cdot (f,x))$ consists of kernels of one-dimensional representations of $C^*(G,\Omega)$, we may use Lemma 7 to get an equivalent sequence (f_n, x_n) converging to (h, y) with each (f_n, x_n) still in the equivalence class of (f, x).

Thus we may assume there is no subsequence of one-dimensional representations.

Note that $\{\psi(g_n \cdot (f_n, x_n)\}_{n=1}^{\infty}$ is a constant sequence of ideals equal to $\psi(h, y)$.

In Lemma 8 we showed that if we had a collection of primitive ideals, $\big\{\{P_n,\}_{n=1}^{\infty}, P\big\}$ of $C^*(G,\Omega)$, with

$$P_n = \ker(\mathrm{ind}_{(G_{x_n},\Omega)}^{(G,\Omega)}(\tau_{f_n,x_n}, M_{x_n})) \longrightarrow P = \ker(\mathrm{ind}_{(G_x,\Omega)}^{(G,\Omega)}(\tau_{f,x}, \rho_x)),$$

that, passing to a subsequence, we could choose:

$$\{y_n\}_{n=1}^{\infty} \subseteq \Omega \text{ and } \{f_n'\}_{n=1}^{\infty} \subseteq \mathfrak{g}^*, \quad \text{with } y_n \to y \in \Omega, f_n' \to f',$$

with $P_n = \ker(\mathrm{ind}_{(G_{y_n},\Omega)}^{(G,\Omega)}(\tau_{f_n',y_n}, M_{y_n}))$ and $P = \ker(\mathrm{ind}_{(G_y,\Omega)}^{(G,\Omega)}(\tau_{f'}, M_y))$. We may also choose the functional-point pairs corresponding to τ_n and τ_n' in the same $\sim_1$ equivalence class as those corresponding to π_n.

Thus by choosing appropriate functional-point pairs in the equivalence class of (f, x) we may realize (h, y) as a subsequential limit, and the equivalence class closures are equal. $\square$

Now to characterize the primitive ideal space of $C^*(G,\Omega)$.

Theorem 4. *The map ψ is a homeomorphism from $\mathfrak{g}^* \times \Omega/\sim$ to Prim$(C^*(G,\Omega))$.*

Proof.

We follow the philosophy of Dana Williams' Theorem 5.3 of [26].

We have the diagram:

$$\begin{array}{ccc} \mathfrak{g}^* \times \Omega & & \\ \big\downarrow q & & \\ \mathfrak{g}^* \times \Omega/\sim & \xrightarrow{\psi} & \operatorname{Prim}(C^*(G, \Omega)) \end{array}$$

As ϕ (Lemma 10) and the natural map q of $\mathfrak{g}^* \times \Omega$ to $\mathfrak{g}^* \times \Omega/\sim$ are continuous, ψ is continuous. By the last lemma ψ is 1-1 onto.

Let F be closed in $\mathfrak{g}^* \times \Omega$ and saturated with respect to $\sim$. We show that $\psi(F)$ is closed in $\operatorname{Prim}(C^*(G, \Omega))$.

Assume $\{P_n\}_{n=1}^{\infty} \subseteq \psi(F)$ and $P_n \to P$. By EH regularity, assume that $P_n = \ker(L_n) = \ker(\operatorname{ind}_{(G_{x_n}, \Omega)}^{(G, \Omega)}(\tau_{f_n, x_n}, M_{x_n}))$.

Pass to a subsequence, apply Lemma 8 to choose $\{f_n'\}_{n=1}^{\infty} \subseteq \mathfrak{g}^*$, $\{y_n\}_{n=1}^{\infty} \subseteq \Omega$, with $f_n' \to f$, $y_n \to x$, and

$$\ker(L_n) = \ker(\operatorname{ind}_{(G_{y_n}, \Omega)}^{(G, \Omega)}(\tau_{f_n', y_n}, M_{y_n})).$$

As $(f_n', y_n) \to (f, x)$ in $\mathfrak{g}^* \times \Omega$, and F is saturated and closed, we are done. $\square$

Section 4

Traces of irreducible representations of $C^*(G,\Omega)$

We give a character theory for $C^*(G,\Omega)$ with G nilpotent Lie, analogous to Kirillov's character theory. Our primary reference is Corwin and Greenleaf [3].

Section 4.1

Schwartz functions on G

Definition 18. On $\mathbb{R}^n$, the Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ are those C^∞ functions f such that

$$\|x^\beta \mathrm{D}^\alpha f\|_\infty < \infty,\ x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n},\ \ \mathrm{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for all multi-indices $\alpha,\beta \in \mathbb{Z}_+^n$. The natural topology of $\mathcal{S}(\mathbb{R}^n)$ is determined by these seminorms. Denote the polynomial coefficient differential operators on $\mathbb{R}^n$ by $\mathcal{P}(\mathbb{R}^n)$, and let $L \in \mathcal{P}$ be arbitrary. We have $f \in \mathcal{S}(\mathbb{R}^n) \iff \|L(f)\|_\infty < \infty$ for all $L \in \mathcal{P}(\mathbb{R}^n)$.

On G, define $\mathcal{S}(G)$, the Schwartz functions on G, to be the functions on G corresponding to $\mathcal{P}(\mathbb{R}^n)$ under a polynomial coordinate map $\psi : \mathbb{R}^n \mapsto G$.

Section 4.2

Traces on $C^*(G)$

We remind the reader some operators on G which are analogues of classical trace operators on finite and compact groups.

For π a unitary infinite-dimensional irreducible representation of a nilpotent Lie group G, the operators $\pi(x)$ have no trace. However, for any $\pi \in \widehat{G}$, there is a tempered distribution θ_π on G that plays the role of the classical trace character $\theta_\pi(g) = \mathrm{Tr}(\pi(g))$ for finite and compact groups. For $\pi \in \mathcal{S}(G)$, the operator

$$\pi(\phi)(\xi) = \int_{s\in G} \phi(s)\pi(s)\,\xi\,\mathrm{d}\mu_G(s) \quad (\xi \in H_\pi)$$

is trace class with a Schwartz kernel K_ϕ; see [3], Theorem 4.2.1.

Definition 19. We may obtain explicit formulas for the kernel integral K_ϕ, once an $f \in \mathfrak{g}^*$, a polarization $\mathfrak{p}$, and a weak Malcev basis ([3], Theorem 1.1.13, pg. 10) through $\mathfrak{p}$ are specified. Let $P = \exp(\mathfrak{p})$; assume $\dim(\mathfrak{g}/\mathfrak{p}) = k$. If $\{X_1, ..., X_n\}$ is the weak Malcev basis, let $p = n - k = \dim(\mathfrak{p})$. Define polynomial maps $\gamma : \mathbb{R}^n \mapsto G,\ \alpha : \mathbb{R}^p \mapsto P,\ \beta : \mathbb{R}^k \mapsto G/P$ by

$$\gamma(s,t) = \exp(s_1X_1)\cdots\exp(s_pX_p)\cdot\exp(t_1X_{p+1})\cdots\exp(t_kX_n)$$

$$\alpha(s) = \gamma(s,0), \quad \beta(t) = \gamma(0,t).$$

Let $\mathrm{d}\mu_{G},\ \mathrm{d}\mu_{M},\ \mathrm{d}\mu_{G/M}$ be the invariant measures on $G, M, G/M$ determined by Lebesgue measures $\mathrm{d}s\,\mathrm{d}t, \mathrm{d}s,\ \mathrm{d}t$, as in Theorems 1.2.10, 1.2.12 and 1.2.13 of [3].

We describe $\mathrm{Tr}(\pi(\phi))$ in terms of integrals over coadjoint orbits in $\mathfrak{g}^*$. Given a Euclidean measure $\mathrm{d}X$ on $\mathfrak{g}$, normalize measures on $\mathfrak{g}$ and $\mathfrak{g}^*$ so that Fourier inversion holds, and define the *Euclidean Fourier transforms* $\widehat{h}$ (resp. $\mathcal{F}h$) of functions h on G (resp. $\mathfrak{g}$), as in [3], pg 137.

Each coadjoint orbit $\mathcal{O}_f = \mathrm{Ad}^*(G)f$ has an invariant measure μ that is unique up to scalar multiple, as $\mathcal{O}_f \cong R_f/G$, where $R_f = \mathrm{Stab}_G(f) = \{x \in G \mid \mathrm{Ad}^*(x)f = f\}$.

Theorem 5. *If π is an irreducible representation of a nilpotent Lie group, corresponding to the co-adjoint orbit $\mathcal{O}_f = Ad^*(G)f \subseteq \mathfrak{g}^*$, there is a unique choice of invariant measure μ on $\mathcal{O}_f$ such that*

$$Tr(\pi(\phi)) = \int_{f\in\mathfrak{g}^*} \widehat{\phi}(f)\mu_f(df) \text{ for all } \phi \in \mathcal{S}(G),$$

with the integral absolutely convergent.

Proof.

See [3], Theorem 4.2.4. □

Section 4.3

Traces of irreducible representations of $C^*(G,\Omega)$

Here we characterize some operators on $C^*(G,\Omega)$ which are trace class. Assume G is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$.

The proof of the next follows by the same logic as Proposition 4.2.2 of [3], observing that we use right actions.

Proposition 4. *Let $\phi \in \mathcal{S}(G)$, $\psi \in C_0(\Omega)$. Let $L = (V, M)$ be an irreducible representation of $C^*(G,\Omega)$ corresponding to $(f,x) \in \mathfrak{g}^* \times \Omega$. Let $k = dim(\mathfrak{g}/\mathfrak{p})$, where $\mathfrak{p}$ is polarizing for the action of f in $\mathfrak{g}_x$. Assume the*

Hilbert space of L is $L^2(\mathbb{R}^k)$. For $\psi \in C_0(\Omega)$ and $\phi \in \mathcal{S}(G)$, then $L(\phi \cdot \psi)$ has kernel K,

$$K(r,s) = \psi(exp(r) \cdot x) \int_{t \in P} \phi(\beta(r)t\beta(s)^{-1}) e^{i \cdot f(\log(t))} d\mu_{\mathfrak{p}}(t)$$

.

Definition 20

Let H be a Hilbert space; we assume that $H \cong L^2(\mathbb{R}^k)$. For a function $\psi \in C_0(\mathbb{R}^k)$, the multiplication operator on H defined by ψ we denote by M_ψ.

Lemma 12. *Let ρ be an irreducible representation of a subgroup H of G, corresponding to the restriction of a functional $f \in \mathfrak{g}^*$ to $\mathfrak{h}$. Defining $\pi = ind_H^G(\rho)$, we have that the the operator $T_{\psi,\phi} = M_\psi \cdot \pi(\phi)$ is trace class with a Schwartz kernel when $\psi \in \mathcal{S}(G/H)$ and $\phi \in \mathcal{S}(G)$. Also, for fixed ψ, $T_{\psi,\phi}$ is tempered in ϕ.*

Proof.

We do this by induction on the codimension of H in G. If $\text{codim}(H) = 0$, this is true by [3], Theorem 4.2.1.

Assume the lemma true for $\text{codim}(H) = n$.

By [3], Theorem 1.1.3 we find a subgroup $G_1 \subseteq G$ with $\text{codim}(G_1) = 1$ and $H \subseteq G_1$. The subgroup H is codimension n in G_1, and G_1 is normal in G by [3], Lemma 1.1.8.

Assume the Lie algebra of G_1 is $\mathfrak{g}_1$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus (\mathbb{R} - \text{span}\{X\})$; we have a smooth cross section for G/G_1 by $\exp(\mathbb{R} \cdot X) \cong \mathbb{R}$.

Define $\tau = \mathrm{ind}_H^{G_1}(\rho)$, acting on the Hilbert space H_τ.

Assume $\pi = \mathrm{ind}_H^G(\rho) \cong \mathrm{ind}_{G_1}^G(\tau)$ acts on the Hilbert space $H_\pi = L^2(\mathbb{R}) \otimes H_\tau$.

Let $a \in G/G_1$, $f \in C^\infty(H_\pi)$. By $f(a)$ we denote the element of H_τ corresponding to $f(a)$. For $z \in G_1$, by $f(a)(z)$ we refer to the value of the H_τ-valued function $f(a)$ in H_τ at the point z in G_1.

Let $\phi_1 \in \mathcal{S}(G/G_1)$, $\phi_2 \in \mathcal{S}(G_1)$, $\psi_1 \in \mathcal{S}(G/G_1)$, and $\psi_2 \in \mathcal{S}(G_1/H)$. Note that sums of elements of the form $\phi_1 \cdot \phi_2$ are dense in $\mathcal{S}(G)$, and sums of the form $\psi_1 \cdot \psi_2$ are dense in $\mathcal{S}(G/H)$.

Let $r \in G/G_1$. Let $f \in C^\infty(H_\pi)$. We have

$$(M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)f)(r) = M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)\pi(t)f(r)\mathrm{d}s\,\mathrm{d}t =$$

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)f(t^{-1}r)\mathrm{d}s\,\mathrm{d}t =$$

(Note that $r^{-1}tr \in G_1$ by normality)

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\pi(s)\tau(r^{-1}tr)f(r)\mathrm{d}s\,\mathrm{d}t =$$

$$M_{\psi_1\cdot\psi_2}\int\limits_{s\in G/G_1}\int\limits_{t\in G_1}\phi_1(s)\phi_2(t)\tau(r^{-1}sts^{-1}r)f(s^{-1}r)\mathrm{d}t\,\mathrm{d}s =$$

(Let $t \to s^{-1}rtr^{-1}s$ in the inner integral and set $s^{-1}r = a$)

$$M_{\psi_1}\int\limits_{a\in G/G_1}\phi_1(ra^{-1})\left[M_{\psi_2}\int\limits_{t\in G_1}\phi_2(ata^{-1})\tau(t)\mathrm{d}t\right]f(a)\mathrm{d}a. \tag{3}$$

Note that for fixed a, by induction hypothesis the operator in the brackets of (3) above is trace class operator on H_τ, and for fixed ψ_2, is tempered in ϕ. Employ [3], Proposition 1.2.8 and for some selection of polynomials $\{P_i\}_{i=1}^n$ and at a fixed point $y \in G_1$ the inner integral of formula (3) equals

$$\left(M_{\psi_2} \int\limits_{t \in G_1} \phi_2\left(P_1(a,t), \ldots P_n(a,t)\right) \tau(t) f(a) \mathrm{d}t \right)(y) = \int\limits_{z \in G_1} k^a_{\phi_2}(z,y) f(a)(z) \mathrm{d}z.$$

By inductive hypothesis, we may find the integral kernel $k^a_{\phi_2}(z,y)$ and it may be chosen Schwartz in y and z.

Recall that for fixed a, $\mathrm{Tr}(k^a_{\phi_2})$ is tempered in ϕ_2 by inductive hypothesis. By [27], Corollary 1, pg. 43 we know that there exists a seminorm ρ on G_1 such and a constant C such that that $|\mathrm{Tr}(k^a_{\phi_2}))| \leq C \cdot \rho(\phi^a_2)$, where ϕ^a_2 is defined as obvious.

Assume that we have multi-indices $\alpha, \beta \in \mathbb{Z}^n$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = \beta_1, \ldots, \beta_n$; define $L = w_1^{\beta_1} \cdots w_n^{\beta_n} \cdot \frac{\partial^{|\alpha|}}{\partial w_1^{\alpha_1} \cdots \partial w_n^{\alpha_i - 1}}$

and we have for the seminorm ρ:

$$|\rho(\phi^a)| = \sup_{w \in \mathbb{R}^n} \left\{ \left| \, L\big(\phi_2\left(P_1(a,w), \ldots, P_n(a,w)\right)\big) \right| \right\} \leq\leq |Q(a)|,$$

by Proposition 1.2.9 of [3] and the n dimensional chain rule, where Q is some polynomial.

So $\mathrm{Tr}(k^a_{\phi_2}) = \int_{y \in G_1} k^a_{\phi_2}(y,y) \mathrm{d}y$ grows no faster than polynomial in a, and $k^a_{\phi_2}$ grows no faster that polynomial in a.

For a in a bounded set, $\phi_2(ata^{-1})$ is bounded by a L^1 function. Let $\mathfrak{p}$ be a polarizing subalgebra of $\mathfrak{g}_x$ with respect to the restriction of f to $\mathfrak{g}_x$. By Proposition 4 we have:

$$k^a_{\phi_2}(r,s) = \psi_2(\exp(r)\cdot x)\int\limits_{t\in P}\phi_2(\beta(r)ata^{-1}\beta(s)^{-1})e^{i\cdot f(\log(t))}\mathrm{d}\mu_{\mathfrak{p}}(t),$$

and we may differentiate in a under the integral sign; infinite differentiability of $k^a_{\phi_2}$ in a follows.

Choose an smooth splitting of G into G/G_1 and G_1 by projections $p : G \mapsto G/G_1$ and $q :\mapsto G_1$,

For our functions ϕ_1, ϕ_2, ψ_1 and ψ_2, define the integral kernel K of $M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)$ by

$$K(x,y) = M_{\psi_1}\big(p(y)\big)\cdot\phi_1\big(p(y)p(x^{-1})\big)\cdot k^{p(y)}_{\phi_2}\big(q(x),q(y)\big).$$

Integrated against $f \in H_\pi$, this gives us $M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)(f)$:

$$(M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)f)(y) = (Kf)(y) = \int\limits_{x\in G} K(x,y)f(x)\mathrm{d}x =$$

$$M_{\psi_1}(p(y))\int\limits_{x\in G}\phi_1(p(y)p(x^{-1}))k^{p(y)}_{\phi_2}(q(x),q(y))f(x)\mathrm{d}x =$$

$$M_{\psi_1}(p(y))\int\limits_{a\in G/G_1}\int\limits_{z\in G_1}\phi_1(p(y)a^{-1})k^a_{\phi_2}(z,q(y))f(a)(z)\mathrm{d}z\,\mathrm{d}a.$$

Note $k^a_{\phi_2}$ is Schwartz on G_1, and its integral and the integrals of its derivitives grow no faster than polynomial in a; henceforth ignore the parameter a.

Note $G/G_1 \cong \mathbb{R}$; treat composition on G/G_1 as addition on $\mathbb{R}$. The functions ψ_1 and ϕ_1 are both Schwartz on G/G_1, and part of the integral kernel is $M_{\psi_1}\big(p(y)\big)\cdot\phi_1\big(p(y)p(x^{-1})\big)$. The other part $k^a_{\phi_2}$ is Schwartz already.

By Peetre's Inquality ([14], pg. 10), when ψ and ϕ are both Schwartz on $\mathbb{R}$, $\psi(y)\phi(y+a)$ is Schwartz on $\mathbb{R}\times\mathbb{R}$.

Similar properties follow for derivitives, and $\psi(y)\phi(y+a)$ is Schwartz on $\mathbb{R}^2$.

Our entire integral kernel is Schwartz, and the operator $M_{\psi_1\cdot\psi_2}\pi(\phi_1\cdot\phi_2)$ is trace class by [3] Theorem A.3.9.

The kernel K is clearly tempered in $\phi_1 \in \mathcal{S}(G/G_1)$; the final result follows by induction and density arguments. $\square$

Henceforth assume that Ω/G is a T_0 space. By Theorem 2.1 [8], (G,Ω) is Polish and for any $x\in\Omega$, we have $G\cdot x \cong G/G_x$.

Definition 21. For any orbit $G\cdot x$, define

$$\mathcal{A}_{G\cdot x} = \mathbb{R}-\text{span}\big\{\phi\cdot\psi \mid \phi\in\mathcal{S}(G),\ \psi\in C_0(\Omega) \text{ and } \psi(\cdot\, x)|_{G/G_x}\in\mathcal{S}(G/G_x)\big\}.$$

Theorem 6. *Let $L=(V,M)$ be the irreducible representation of $C^*(G,\Omega)$ associated to the pair $(f,x)\in\mathfrak{g}^*\times\Omega$. The representation L is trace class on $\mathcal{A}_{G\cdot x}$, and for fixed ψ, is tempered in ϕ.*

Proof.

Note $L(\psi\cdot\phi) = M_\psi(\cdot\, x)\cdot V(\phi)$, and apply Lemma 12 just proven. $\square$

Definition 22. Remember Definition 19, where we defined maps α, β, γ of $\mathfrak{p}$, $\mathfrak{g}/\mathfrak{p}$, and $\mathfrak{g}$ (resp.) to G. Assume $f\in\mathfrak{g}^*$, and $\mathfrak{g}_x$ is the Lie algebra of G_x for $x\in\Omega$. Assume $\dim(\mathfrak{g}_x)=l$. Define a new map $\delta:\mathbb{R}^l\mapsto G_x$ by $\delta(s)=\gamma(s_1,\ldots,s_l,0)$.

Theorem 7. *Let L be a representation of $C^*(G,\Omega)$ corresponding to the functional-point pair $(f,x) \in \mathfrak{g}^* \times \Omega$.*

Let $\mathfrak{p}$ be a polarizing subalgebra of $\mathfrak{g}_x$ with respect to $f|_{\mathfrak{g}_x}$; assume $k = dim(\mathfrak{g}/\mathfrak{p})$.

Assume L acts on the Hilbert space $L^2(\mathbb{R}^k)$.

Let $\mathcal{O}_L$ be the $\sim_1$ equivalence class of (f,x) in $\mathfrak{g}^* \times \Omega$, specifically,

$$\mathcal{O}_{(f,x)} = \{(l,y) \in \mathfrak{g}^* \times \Omega \mid \text{for some } s \in G, \text{we have}$$
$$l = Ad^*(s)(f+h),\ y = s \cdot x,\ h \in \mathfrak{g}_x^{\perp}\}.$$

Let p, q be the natural projections from $\mathcal{O}_L$ to $\mathfrak{g}^*$ and Ω, respectively. Let $\phi \in \mathcal{S}(G)$, $\psi(\cdot\ x) \in \mathcal{S}(G/G_x)$. We have

$$Tr(L(\phi \cdot \psi)) = \int_{z \in \mathcal{O}_L} \psi(q(z))\widehat{\phi}(p(z))\, dz$$

for a particular choice of G-invariant measure dz on $\mathcal{O}_L$.

Proof.

We closely mimic the proof of Theorem 4.2.4 on pp. 138-41 of Corwin and Greenleaf [3].

Assume $\mathfrak{p}$ has dimension p, so $n = \dim(\mathfrak{g}) = p + k$. As G_x is fixed, denote it by S, and its Lie algebra by $\mathfrak{s}$.

Give $\mathfrak{g}$ a weak Malcev basis (again, [3], Theorem 1.1.13, pg. 10) through $\mathfrak{p}$ and $\mathfrak{s}$. Assume that L acts on $L^2(\mathbb{R}^k)$. By Theorem 6, Proposition 4, and Theorem A.3.9 of [3],

$$\mathrm{Tr}(L) = \int_{s \in \mathfrak{g}/\mathfrak{p}} K(s,s)\mathrm{d}s =$$

$$\int_{s\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(s)\cdot x) \int_{t\in\mathfrak{p}} \phi(\beta(s)\mathrm{exp}(t)\beta(s)^{-1})e^{i\cdot f(t)}\mathrm{d}\mu_{\mathfrak{p}}(t)\,\mathrm{d}\mu_{\mathfrak{g}/\mathfrak{p}}(s).$$

Let $\mathfrak{z}$ be a subspace complementary to $\mathfrak{p}$ in $\mathfrak{g}$; take $\mathfrak{z} = \mathbb{R}$-span of the last k basis vectors of our Malcev basis through $\mathfrak{p}$. We have an additive splitting $H + X \in \mathfrak{p} \oplus \mathfrak{z}$ for each element in $\mathfrak{g}$.

Let $\mathrm{d}X$, $\mathrm{d}H$ be arbitrarily assigned Euclidean measures on $\mathfrak{z}$, $\mathfrak{p}$; we have that $\mathrm{d}H\,\mathrm{d}X$ is a Euclidean measure on $\mathfrak{g}$, which we use to define the above integrals.

For $\phi \in \mathcal{S}(G)$, let $u \in \mathbb{R}^k \cong \mathfrak{z} \cong \mathfrak{g}/\mathfrak{p}$, and following [3], middle of pg. 139, define

$$\Phi_u(H,X) = \phi(\beta(u)\mathrm{exp}(H+X)\beta(u)^{-1}).$$

Take the above two displayed equations, and follow the proof of [3] Theorem 4.2.4, up to formula (20) at the bottom of pg. 140, remembering that we use right actions.

We get

$$(4)\quad \mathrm{Tr}(L(\psi\cdot\phi)) = \int_{u\in\mathfrak{g}/\mathfrak{p}} \psi(\beta(u)\cdot x) \int_{f^\perp\in\mathfrak{p}^\perp} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u)^{-1})(f+f^\perp)\big)\mathrm{d}f^\perp\,\mathrm{d}u.$$

Split $\mathfrak{g}/\mathfrak{p}$ into $\mathfrak{g}/\mathfrak{s}$ and $\mathfrak{s}/\mathfrak{p}$. As $\beta(t)\cdot x = x$ for all $t \in \mathfrak{s}/\mathfrak{p}$, formula (4) above equals

$$\int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{v\in\mathfrak{s}/\mathfrak{p}} \int_{f^\perp\in\mathfrak{p}^\perp} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\beta(v))(f+f^\perp)\big)\mathrm{d}f^\perp\,\mathrm{d}v\,\mathrm{d}u =$$

(5)

$$\int\limits_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int\limits_{v\in\mathfrak{s}/\mathfrak{p}} \int\limits_{f_2^\perp\in\mathfrak{s}^\perp} \int\limits_{f_1^\perp\in\mathfrak{p}^\perp/\mathfrak{s}^\perp} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\beta(v))(f+f_1^\perp+f_2^\perp)\big) \, \mathrm{d}f_1^\perp \, \mathrm{d}f_2^\perp \, \mathrm{d}v \, \mathrm{d}u.$$

Assume that f is nonzero only on $\mathfrak{s}$.

Now work with the innermost integral in $f_1^\perp$. If R_f is the stabilizer of the functional $f|_\mathfrak{s}$, there exists invariant measures $\mathrm{d}\dot{p}$ and $\mathrm{d}\dot{x}$ on P/R_f and S/P such that $\mathrm{d}\dot{p}\,\mathrm{d}\dot{x}$ is invariant measure on S/R_f.

By Proposition 3.1.18 on pg. 97 [3], $\mathrm{Ad}^*(P)(f) = f + \mathfrak{p}^\perp/\mathfrak{s}^\perp = (f + \mathfrak{p}^\perp)|_\mathfrak{s}$. The natural diffeomorphism $\Delta : P/R_f \mapsto \mathrm{Ad}^*(P)(f) = (f+\mathfrak{p}^\perp)|_\mathfrak{s}$ is equivariant and measure preserving on $(f+\mathfrak{p}^\perp)|_\mathfrak{s}$. On the other hand, we may transfer Euclidean measure $\mathrm{d}f_1^\perp$ on $\mathfrak{p}^\perp/\mathfrak{s}^\perp$ under the translation map q, where $q(f_1^\perp) = f + f_1^\perp$, to a Euclidean measure $\nu = q^*(\mathrm{d}f_1^\perp)$ on the affine space $f + \mathfrak{p}^\perp/\mathfrak{s}^\perp$.

Note $(\mathrm{Ad}^*(p)f - f) \in \mathfrak{p}^\perp/\mathfrak{s}^\perp$; see again Proposition 3.1.18 on pg. 97 of [3].

For each $p \in P$, define an affine map $A(p)$ from $\mathfrak{p}^\perp/\mathfrak{s}^\perp$ to itself by $A(p)(f_1^\perp) = \mathrm{Ad}^*(p)f_1^\perp + (\mathrm{Ad}^*(p)f - f)$, and

$$q \circ A(p)(f_1^\perp) = q\big(\mathrm{Ad}^*(p)(f_1^\perp) + (\mathrm{Ad}^*(p)f - f)\big) = \mathrm{Ad}^*(p)(f + f_1^\perp).$$

As the linear part $\mathrm{Ad}^*(p)|_{\mathfrak{p}^\perp/\mathfrak{s}^\perp}$ of $A(p)$ is unipotent, the operator $A(p)$ preserves $\mathrm{d}f_1^\perp$, and $\mathrm{Ad}^*(p)$ preserves ν on the affine space $f+\mathfrak{p}^\perp/\mathfrak{s}^\perp$. As $\mathrm{Ad}^*(p)$

is also measure preserving on P/R_f, under Δ, ν is identified with an invariant measure on P/R_f, which must be a scalar multiple of $\mathrm{d}\dot{p}$: $(\Delta^{-1})^*\nu = c\cdot \mathrm{d}\dot{p}$. Hence if $\psi \in L^1(f+\mathfrak{p}^\perp/\mathfrak{s}^\perp)$, we must have

$$\int_{f_1^\perp \in \mathfrak{p}^\perp/\mathfrak{s}^\perp} \psi(f+f_1^\perp)\mathrm{d}f_1^\perp = \int_{f'\in f+\mathfrak{p}^\perp/\mathfrak{s}^\perp} \psi(f')\mathrm{d}\nu(f') = c\cdot \int_{\dot{p}\in P/R_f} \psi(\mathrm{Ad}^*(\dot{p})f)\mathrm{d}\dot{p}.$$

Consequently, formula (4) above equals

$$\int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{v\in\mathfrak{s}/\mathfrak{p}} \int_{f_2^\perp\in\mathfrak{s}^\perp} c\cdot \int_{m\in\mathfrak{p}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(v))(\mathrm{Ad}^*(\alpha(m))f + f_2^\perp)\big) \, \mathrm{d}m\,\mathrm{d}f_2^\perp\,\mathrm{d}v\,\mathrm{d}u =$$

(by Haar measure, move the $\mathrm{Ad}^*(\alpha(m))$ to include $f_2^\perp$)

$$c\cdot \int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{v\in\mathfrak{s}/\mathfrak{p}} \int_{f_2^\perp\in\mathfrak{s}^\perp} \int_{m\in\mathfrak{p}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(v))(\mathrm{Ad}^*(\alpha(m))(f + f_2^\perp))\big) \, \mathrm{d}m\,\mathrm{d}f_2^\perp\,\mathrm{d}v\,\mathrm{d}u =$$

(combining v and m into a single variable y)

$$c\cdot \int_{u\in\mathfrak{g}/\mathfrak{s}} \psi(\beta(u)\cdot x) \int_{f_2^\perp\in\mathfrak{s}^\perp} \int_{y\in\mathfrak{s}/\mathfrak{r}_f} \widehat{\phi}\big(\mathrm{Ad}^*(\beta(u))\mathrm{Ad}^*(\delta(y))(f+f_2^\perp)\big)\mathrm{d}y\,\mathrm{d}f_2\,\mathrm{d}u.$$

This is our orbital integral. $\square$

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References

1. I. Brown, *Dual topology of a nilpotent Lie group*, Ann. scient. Éc. Norm. Sup. **6** (1973), 407-411.
2. R.C. Busby, *Double centralizers and extensions of C^* algebras*, Trans. Amer. Math. Soc. **132** (1968), 79-99.
3. L. Corwin and F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications part 1*, Cambridge University Press, Cambridge, 1990.
4. ______, *Representations of nilpotent Lie groups and their applications, part 2, preprint.*
5. J. Dixmier, *C^* Algebras*, North-Holland Publishing Co., Amsterdam, New York, Oxford, 1977.
6. S. Doplicher, D.Castler, and D.W.Robinson, *Covariance algebras in field theory and statistical mechanics*, Comm. Math. Phys. **3** (1966), 1-28.
7. S. Echterhoff, *The primitive ideal space of twisted covariant systems with continuously varying stabilizers*, Math Ann. **292** (1992).
8. E. Effros, *Tranformation groups and C^* algebras*, Ann. Math. **81** (1965), 38-55.
9. E. Effros and F. Hahn, *Locally compact transformation groups and C^* algebras*, Mem. Amer. Math. Soc. **75** (1967).
10. J.M.G. Fell, *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space*, Proceedings of the American Mathematical Society **13** (1962).
11. ______, *The dual spaces of C^* algebras*, Trans. Amer. Math. Soc. **94** (1960), 365-403.
12. ______, *Weak containment and induced representations of groups*, Canadian J. Math **14** (1962), 237-268.
13. ______, *Weak containment and induced representations of groups II*, Trans. of the AMS **110** (1964), 424-447.
14. P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish Press, Wilminton, DE, USA, 1984.
15. J. Glimm, *Families of induced representations*, Pac. J. Math **12** (1962), 885-911.
16. E. Gootman, *The type of some C^* and W^* algebras associated with transformation groups*, Pac. J. Math. **48, no 1** (1973), 98-106.
17. E. Gootman and J. Rosenberg, *The structure of crossed product C^* algebras: a proof of the generalized Effros-Hahn conjecture*, Invent. Math. **52** (1979), 283-298.
18. P. Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191-250.
19. K. Joy, *A description of the topology on the dual space of a nilpotent Lie group*, Pac. J. Math **12, no 1** (1984), 135-139.

20. J. Kelley, *General Topology*, Van Nostrand co., New York, Toronto, London, Melbourne, 1955.

21. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russ. Math. Survey **17** (1962), 53-104.

22. G. Mackey, *Induced representations of locally compact groups I*, Ann. of Math. **55, no.1** (1952), 101-139.

23. ______, *The theory of unitary group representations*, University of Chicago Press, Chicago and London, 1976.

24. G. Murphy, *C^* algebras and operator theory*, Academic Press, Boston San Diego New York London Sydney Tokyo Toronto, 1990.

25. M. Takesaki, *Covariant representations of C^* algebras and their locally compact automorphism groups*, Acta Math. **119** (1967), 273-303.

26. D.P. Williams, *The topology on the primitive ideal space of transformation group C^* algebras and C.C.R. algebras*, Trans. Am. Math. Soc. **266, no 2** (1981), 335-359.

27. K. Yosida, *Functional Analysis, (third edition)*, Springer-Verlag, Berlin Heidelberg New York, 1980.